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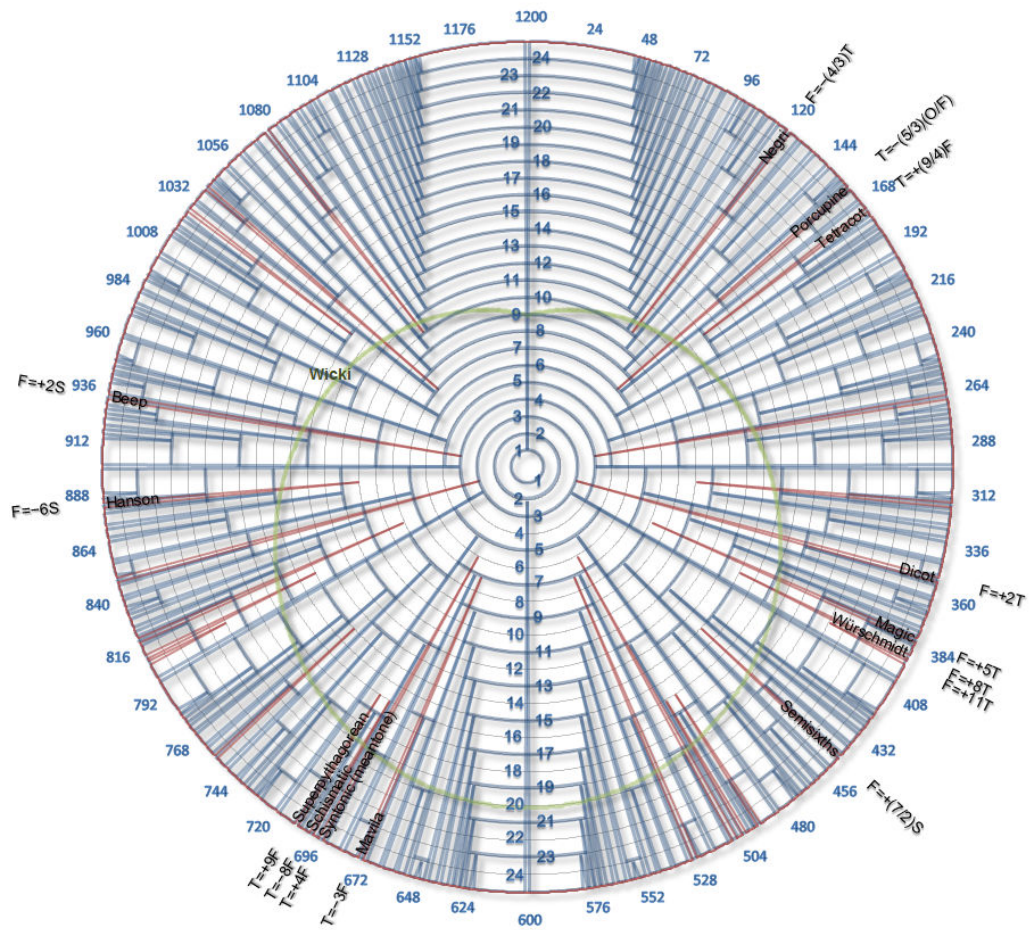
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X_System

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November, 2006



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1 Introducing the X_System

The X_System makes the playing, writing, and learning of music – even when using unconventional tunings – more intuitive, more logical, more expressive, and better sounding.

The X_System allows for:

- different temperaments to be chosen at the flick of a switch;
- tunings to be dynamically altered at the push of a lever;
- the use of a special hexagonal button-field that allows for any given interval or chord always to have the same shape on that button-field;
- consonant chords to have their consonance maximised, whatever the tuning actually chosen;
- radical changes to be made to the timbral character of tones using a minimal number of controls;
- a choice of keyboard mappings, which enable for the balance between number of intervals and octaves to be altered.

The X_System is designed for both the beginner and the expert. The above features make it easier for the beginner to understand the basics of music, but the logic, breadth and depth with which they are applied, opens up unparalleled levels of control and experimentation for the musical expert.

The X_System integrates the following X_Elements into a coherent system:

- X_Temperament
- X_Tuning
- X_Spectrum
- X_Timbre
- X_Tonality
- X_Scale
- X_Layout

The theory of the X_System is called X_Theory, and this paper is the definitive X_Theory of the X_System.

2 X_Temperaments and X_Tunings

A regular temperament is a specific type of retuning (tempering) of a system of just intonation. The purpose of this tempering is to remove unwanted commas found in the just intonation, and/or to simplify the just intonation so that it can be more easily mapped to an instrument or notational system. For a temperament to be deemed successful or appropriate, it must not adversely affect the tuning of specific intervals.

An *X_Temperament* comprises a simple tuning system, a mapping from a just intonation to that tuning system, a tuning range over which the temperament can be considered valid, and a means of expressing these elements such that they function as the nexus of the X_System. The means by which these elements are formulated and presented to the artist is the subject of this section.

In order to ground the discussion, Section o begins by stating the underlying assumptions and definitions on which the X_System is based.

2.1 Temperaments and Tunings

Definition: A *tuning system* is a set of intervals used for a musical purpose.

Definition: A *regular tuning system* is a set of intervals such that each is obtainable as a product of integral powers of a finite number of intervals called *generators*.

Definition: Two or more intervals a_1, a_2, \dots, a_n , are *multiplicatively dependent* if there are integers z_1, z_2, \dots, z_n , not all zero, such that $a_1^{z_1} * a_2^{z_2} * \dots * a_n^{z_n} = 1$. If there are no such z_1, z_2, \dots, z_n , then a_1, a_2, \dots, a_n , are said to be *multiplicatively independent*.

Definition: The *rank* of a regular tuning system is equivalent to the number of multiplicatively independent generators it requires.

Definition: Any group of multiplicatively independent generators that is used to generate a regular tuning system is called a *generating set*.

For example, a rank-2 regular tuning system requires a generating set of two multiplicatively independent intervals, a rank-4 regular tuning system requires a generating set of 4 multiplicatively independent intervals.

Definition: A *regular tuning* is a specific realisation of a given regular tuning system – i.e. one in which the tuning of the generators has been defined.

Note, however, that any given regular tuning can be generated by an infinite number of different generating sets, so a regular tuning is not uniquely defined by a generating set.² When choosing between different possible generating sets, it is often convenient to choose one that contains familiar (i.e. consonant) octave-reduced intervals.

² If the elements of the generating set A are expressed as a_1, a_2, \dots, a_n , and the elements of generating set B are expressed as $a_1^{p_1} * a_2^{q_1} * \dots * a_n^{i_1}, a_1^{p_2} * a_2^{q_2} * \dots * a_n^{i_2}, \dots, a_1^{p_n} * a_2^{q_n} * \dots * a_n^{i_n}$, then if matrix $\begin{bmatrix} p_1 & \dots & p_n \\ \vdots & \ddots & \vdots \\ i_1 & \dots & i_n \end{bmatrix}$ has a determinant of ± 1 , sets A and B produce identical tuning systems.

Definition: A *p-limit just intonation* is a regular tuning system whose generators are all of the primes up to a given prime p .

For example, the generators of 5-limit just intonation are 2, 3, and 5 (which can be re-expressed using the more familiar octave-reduced generating intervals $2/1$, $3/2$, $5/4$).

The fundamental theorem of arithmetic states that every positive integer greater than 1 can be expressed in exactly one way, apart from rearrangement, as a product of one or more primes. This means that no just intonation interval that is generated by a particular set of primes can be generated by a different set of primes.

Definition: A *p-limit regular temperament* is a specific mapping of a p -limit just intonation to a regular tuning system with a lower rank.

Due to the fundamental theorem of arithmetic, such mappings can only be achieved if at least some of the just intonation intervals (which are, by definition, generated by primes) are tempered (i.e. mistuned).³

There are an infinite number of possible mappings from a just intonation to a lower rank tuning system. The function used to achieve a specific mapping can be represented by the intervals (usually termed *commas*) that are tempered to unison in that regular temperament. The *kernel* of a regular temperament is the group of just intonation commas that are mapped to unison in that temperament. If a just intonation has rank m (the number of prime generators it contains), the tuning system to which that just intonation is mapped has rank r (the number of multiplicatively independent generators it requires), and the kernel has rank k (the number of multiplicatively independent commas it contains), then $\text{rank } m = \text{rank } k + \text{rank } r$. This means, for example, that to map an 11-limit just intonation (which is rank-5) to a rank-2 tuning system requires 3 multiplicatively independent commas to be tempered to unison.

A regular temperament is not uniquely defined by any particular set of commas, because different sets of commas can define the same temperament. For example, in the meantone temperament, the syntonic comma ($81/80 = 2^{-4}3^45^{-1}$) is tempered to unison, which means that every integral power of the syntonic comma (e.g. $(81/80)^2$ or $(81/80)^{-7}$) is also tempered to unity, so any of these infinite number of different commas could be used to define the temperament. Similarly, in a temperament that tempers to unity more than one multiplicatively independent commas, every possible product of integral multiples of these commas is also tempered to unison. For this reason, it makes sense to define a temperament with those commas that are “simpler”; for example, in the case of meantone it would be pointless to describe the temperament as being that which tempers out $6561/6400$ (which is $(81/80)^2$) instead of $81/80$. When there are two or more commas, however, the choice is more ambiguous. It is possible, however, to uniquely determine a *comma sequence* such that it does uniquely define a temperament (Smith). One advantage of using commas to describe regular temperaments is that they help to indicate the structural properties of the resulting temperament (see Section 2.2.1).

³ The number of prime intervals of different prime-limit that can be justly tuned simultaneously, is the same as the rank of the tuning system that hosts the temperament.

Every different regular temperament requires different tuning compromises of different magnitudes. This is illustrated by the following examples of rank-2 5-limit regular temperaments – i.e. regular temperaments where 5-limit just intonation (which is rank-3) is mapped to a rank-2 tuning system by representing both 5-limit intervals and 3-limit intervals with at least one common generator:

- in the meantone temperament, the just major third is mapped to (i.e. represented by) four perfect fifths minus two octaves (the common generator being the perfect fifth);
- in the “schismatic” temperament, the just major third is mapped to (i.e. represented by) five octaves minus eight perfect fifths (the common generator being the perfect fifth);
- in the “magic” temperament, the perfect fifth is mapped to (i.e. represented by) five major thirds minus one octave (the common generator being the major third);
- In the “porcupine” temperament, the major third is mapped to one octave minus five thirds of a perfect fourth (the common generator being the third-of-a-fourth).

The tuning compromises required for each of the first three temperaments above are illustrated in Figure 1 to .

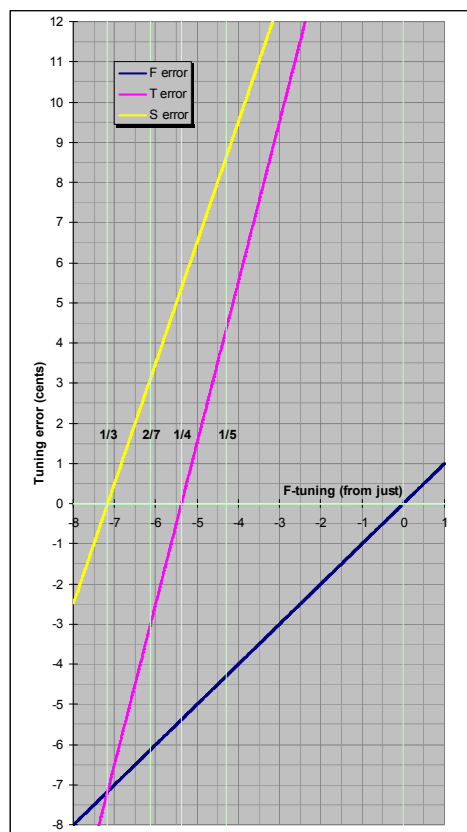


Figure 1. Meantone temperament tuning errors – the perfect fifth (F) is usually flattened by a few cents to produce a good sounding major third (T) and major sixth (S).

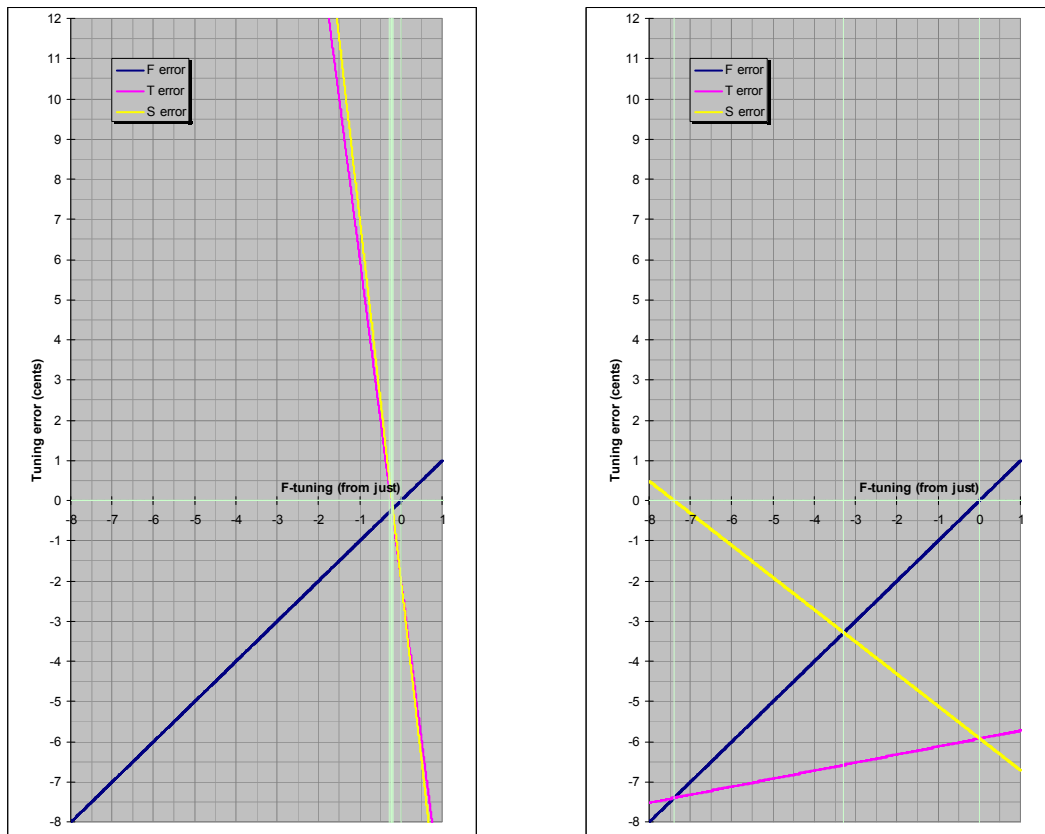


Figure 2. On the left: Schismatic temperament tuning errors – the perfect fifth requires hardly any tempering to produce very pure sounding thirds and sixths. On the right: Magic temperament tuning errors – the major third is usually flattened by a few cents to give good sounding perfect fifths and fourths.

A regular temperament is, therefore, a convenient fiction that rests upon a cognitive “suspension of disbelief” – a masquerade of intervals whose true identity is revealed only if the temperament requires, or is realised with, too great a mistuning.

For a mapping to have psychoacoustic as well as mathematical validity, therefore, it must be possible for the tempered intervals to *signify* their untempered (i.e. just) referents. If the temperament produces intervals that do not signify their just intonation referents, then that temperament has no psychoacoustic foundation. This implies that the tempered versions of the intervals need to be reasonably close in pitch ratio to their just referents, though pitch ratio is not the only means of signification.

Definition: The *valid tuning range* of a temperament is the arbitrary range over which it can be considered to be psychoacoustically valid as a specific mapping of just intervals to tempered intervals.

Methods of defining and implementing valid tuning ranges are discussed in Section 2.4.

Definition: The *unit of repetition* is an interval that represents the equivalence interval. For tones with harmonic spectra, this interval is typically the octave O with ratio $2/1$.

When using standard musical tones, which have spectra comprising harmonic partials, the *affinity* of tones separated by an octave is so pronounced that they are considered to be equivalent. “Affinity means that tones may be perceived as similar in certain aspects; that in some respect a tone may be replaced by another one; and that one tone may even be confused with another one” (Terhardt).

Definition: Any interval t may be written as sO^m where $m = \text{floor}(\log_O(t))$ is the integer such that $1 \leq s < O$. The interval s is said to be the *unit-reduced* (or, *octave-reduced*) version of t .

Definition: The *cardinality* of a tuning system, or any subset of that tuning system (such as a scale), is the number of octave reduced notes it contains.

Definition: An *s-chain* of cardinality N is the complete set of N notes generated by the powers $n = 0$ to $N - 1$ of an interval s , and then octave reduced.

Observe that an *s-chain* requires a regular tuning system with rank at least two, where one of the generators is an integral root of an octave.

Definition: An *s-tuning* is a specific realisation of a given *s-chain*.

Every *p-limit* just intonation has a major prime chord and a minor prime chord that, along with their octave inversions, contain all of the intervals that are considered consonant in that tuning system.

Definition: The *major prime chord* of a *p-limit* just intonation is built as an octave-reduced version of $2:3:5:\dots:p$. In 3-limit, the major prime chord is 2:3; in 5-limit, the major prime chord is 4:5:6; in 7-limit, the major prime chord is 4:5:6:7; in 11-limit, the major prime chord is 8:10:11:12:14; in 13-limit, the major prime chord is 8:10:11:12:13:14; etc..

Definition: The *minor prime chord* of a *p-limit* just intonation is built as an octave-reduced version of $1/2:1/3:1/5:\dots:1/p$. In 3-limit, the minor prime chord is 2:3; in 5-limit, the minor prime chord is 10:12:15; in 7-limit, the minor prime chord is 60:70:84:105; in 11-limit, the minor prime chord is 660:770:840:924:1155; in 13-limit, the minor prime chord is 8580:9240:10010:10920:12012:15015; etc..

Definition: The *primary consonances* of any *p-limit* just intonation are the just octave, all of the intervals found in the prime chords, and their octave inversions.

In 5-limit just intonation, the primary consonances are $jO = 2/1$, $jF = 3/2$, $jT = 5/4$, $jS = 5/3$, and their octave inversions $jO/jF = 4/3$, $jO/jT = 8/5$, $jO/jS = 6/5$. Note that $jS = jO jF^{-1} jT$. The tempered versions of these intervals are O , F , T , and S , respectively, and it is assumed that $S = O F^{-1} T$. The octave inversions of each of these intervals can be ignored for the purposes of the proceeding calculations because, assuming $O = 2$, when the tuning of any interval is pure so is its octave inversion.

Definition: In higher-limit just intonations and temperaments, the following symbols are used to represent the following primary consonances: $jD = 7/4$ (D is tempered version); $jL = 11/8$ (L is tempered version); $jH = 13/8$ (H is the tempered version); $jV = 17/16$ (V is the tempered version); $jN = 19/16$ (N is the tempered version).

For an interval with ratio p/q (expressed in its lowest terms so that p and q are co-prime), a rough guide to consonance and discriminability (Vos and van Vianen) (assuming harmonic spectra) can be given by $2 / (p + q)$, which gives the “density” of coincident harmonics compared to all harmonics. Vos and van Vianen (1984) show how discriminability of intervals by frequency difference is proportional to $p + q$ – the equation $2 / p + q$ is a normalisation of this so that the maximally consonant interval (the unison) has a value of 1.

2.2 Why Temper?

Just intonation provides “perfectly” tuned intervals that are maximally consonant and perfectly “in tune”; so why would anyone choose to temper this perfection?

2.2.1 Disruptive Commas

Any non-primary consonance which is formed by the addition or subtraction of various primary consonances is called a *secondary interval*. Despite their non-primary status, some of these secondary intervals may still be considered to serve a useful melodic, or even harmonic, function within a given musical system; conversely, those secondary intervals that do not serve any function, within a given musical system, or are disruptive to that musical system, are termed *commas*.

In conventional common practice music, the secondary intervals are: the major second (which can be found between the perfect fourth and perfect fifth); the minor second (which can be found between the major third and perfect fourth); the augmented unison (which can be found between the minor third and major third); and their inversions. In the common practice musical system, these secondary intervals are considered useful. All other secondary intervals, such as the syntonic comma (found between two major seconds and a major third), the Pythagorean comma (found between twelve perfect fifths and unison), and the enharmonic diesis (found between an augmented second and a minor third), are not considered useful.

Indeed, some commas are disruptive, antithetical even, to the functionality of the chosen musical system. An example of this is the syntonic comma which disrupts the smooth functioning of common practice tonality. In just intonation, a simple cadential chord progression such as I – vi – ii – V – I requires commatic drift or shift so that the final I is at a slightly different pitch to the starting I (drift), or that one of the notes has to change its pitch slightly between two chords (shift). Similarly the enharmonic diesis is invoked by enharmonic modulation, so a chord progression such as i – iv – $\flat V$ aug6 – $\flat II$ – i will, strictly speaking, also require commatic shift or drift. Any comma that can be invoked by a functional progression within a particular musical system is termed a *disruptive comma*. The importance of the functional progression(s) within which a comma is invoked, determines how disruptive that comma is. In common practice tonality, the syntonic comma is invoked in *every* full cadential progression – this makes it a profoundly disruptive comma; the enharmonic diesis is only invoked by more advanced and less often used progressions, and is always part of a dissonant context, so is less disruptive (this explains why meantone temperaments that temper out the syntonic comma but not the enharmonic diesis, are perfectly acceptable for much of the Western repertoire).

One of the principal purposes for a temperament is to temper out disruptive commas. Indeed, the process of tempering and the negation of commas has enhanced the

functionality of Western music: allowing for the major third to be equated to two major seconds has enabled functional tonal cadences; allowing for thirds to be equated to augmented seconds and diminished fourths has expanded tonal functionality by allowing for enharmonic substitution and the profound modulations and harmonic substitutions that this allows; allowing for twelve fifths to be equated to unison has allowed for music to cycle by fifths through twelve keys but still return to its starting point (for example, Bach's *Well-Tempered Clavier* and Chopin's *Preludes*).

Common-practice may well not be the only musical system capable of producing a unified tonal functionality. Any alternative system will have its own disruptive commas, which will require a temperament that tempers them out.

2.2.2 Rank and “Playability”

For a tuning system to be mapped to an instrument whose notes are played with keys, pads, buttons, etc., so that every interval within that tuning system has a unique representation on that keyboard or button-field, requires that the tuning system has a rank no higher than the number of dimensions within which the button-field or keyboard is spatially arranged (see *X_Layout* Section 6, for more information). Such a mapping is called an *isomorphic* mapping.

With a view to practicability, any keyed instrument can be constructed with no more than two dimensions. This means that for an isomorphic mapping, a tuning system of rank no higher than two can be used. This means that 3-limit is the highest-limit just intonation that can be isomorphically mapped to a button-field. Higher-than-3-limit intonations must, therefore, be tempered to a rank-2 tuning system for them to have an isomorphic mapping to a button-field.

2.2.3 Rank and “Notate-ability”

For any piece of music to be easily and quickly readable, it is required that it consists of relatively few symbols. The lower the rank of the tuning system the simpler the notational system can be.

For instance, the rank-2 tuning system assumed for common practice notation assigns a separate note name, number, or staff position for only seven notes out of the entire gamut – a, b, c, d, e, f, and g (or 1, 2, 3, 4, 5, 6, 7). All other notes or intervals are notated through the use of context dependent modifiers such as “#”, “x”, “sharp”, “double sharp”, “augmented”, “double augmented”, “b”, “bb”, “flat”, “double flat”, “diminished”, “double diminished”, etc.. Cascading these modifiers, allows for any possible note or interval within the temperament to be notated – e.g. “f#####”, or “sextuple augmented fourth”.

Higher-rank tuning systems need further symbol sets (one for each rank) to allow for an unlimited notational system.

2.3 Mapping Just Intonations to X_Generating Sets

A regular temperament is a specific means to map a just intonation to a lower rank regular tuning system. Since any given regular tuning can be generated by an infinite number of possible generating sets, it is necessary to choose a generating set optimised for the *X_System*'s requirements.

An optimised X_Generating Set must be capable of being mapped onto a 2-dimensional button lattice, and it preferably uses intervals that can be quickly recognised or understood by the artist. It also contains a means of indicating exactly how any given temperament maps just intonation intervals to those generators. It is easier to understand octave-reduced intervals, due to their greater familiarity, and the primary consonances found in any given temperament provide the ideal resource from which to choose suitable generating intervals.

An X_Generating Set has the following properties:

- a rank-2 generating set – this is necessary to enable the temperament to be isomorphically mapped to a 2-dimensional button-field;
- a unit of repetition generator $s(U)$ that is the lowest possible root of O – it is desirable that one of the generators is as simply related as possible to this important interval of equivalence;
- a second generator $s(G)$ that is chosen according to the following priority: firstly, that it is an integral root of the lowest possible limit primary consonance; secondly, that that integral root is as low as possible; thirdly, that it has the highest possible $2 / (p + q)$ value – this prioritisation ensures that the generating interval is as simply related as possible to the most familiar and consonant interval available.
- further redundant (multiplicatively dependent) generators $s(R)$ that succinctly indicate how all primary consonances within that temperament are generated by $s(U)$ and $s(G)$; and which, therefore, uniquely define that temperament and differentiate it from all other temperaments that share the same tuning system generators.

A redundant generator is required to define the mapping for each prime up to the prime limit, except for those directly represented by the non-redundant generators; so a 5-limit temperament requires one redundant generator, a 7-limit temperament requires two, an 11-limit temperament requires three, a 13-limit temperament requires four, and so on.

A redundant generator can be expressed in a *complex form* and a *simple form*. In the complex form the equalities are expressed as powers (e.g. $T = O^{-2} F^4$), including the octaves needed for reduction. This form is favoured when included as part of a full generating set, or when the extra detail is required. In the simple form, the equalities are expressed as if the frequencies are logarithmic (and so become additive) (e.g. $T = +4F$), and unit-reduction is not included unless the unit of repetition is a root of the octave (e.g. $T = -\frac{1}{4}O + 1F$).

The simple form is easier to understand – the sign and numerator of the numbers before the generating interval(s) shows how many steps up or down the s -chain (and/or unit of repetition chain) the derived interval is found, the denominator (when present) of the number before the generating interval(s) shows what division of the generating interval one step along the s -chain (or unit of repetition chain) represents. For example, $T = +9/4F$ means that the interval representing $5/4$ is found by going up 9 s -chain steps, each of which represents one quarter of a $3/2$ (ignoring octaves).

This is demonstrated in the following figures, which show the intervals of the redundant generators as played on the Thummer keyboard using its default mapping.

The default mapping places the generating interval up one row then right to the next button, and the unit of repetition is reached by going up two buttons. The steps along the s -chain are indicated by the numbers, the positions of the primary consonances F , O/F , T , D , etc. are shown underneath the s -chain step number. Note how the redundant generator quickly indicates the how the primary consonances are derived from the generator:

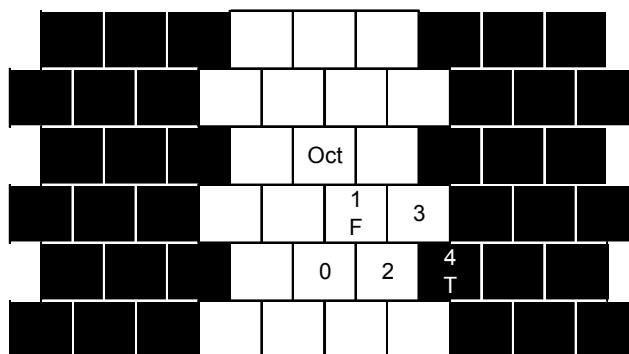


Figure 3. $T = +4F$, or $T = O^{-2} F^4$, meantone.

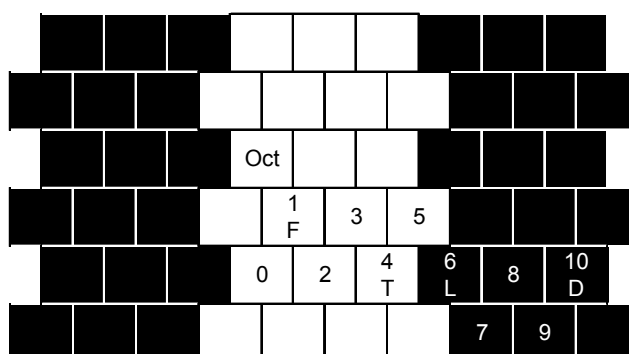


Figure 4. $T = +4F$, $D = +10F$, $L = +6F$.

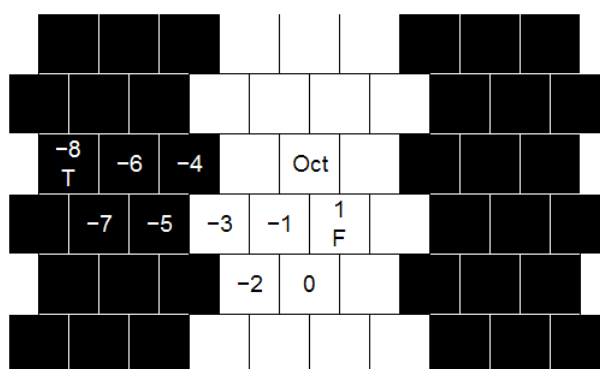


Figure 5. $T = -8F$, or $T = O^5 F^{-8}$, schismatic.

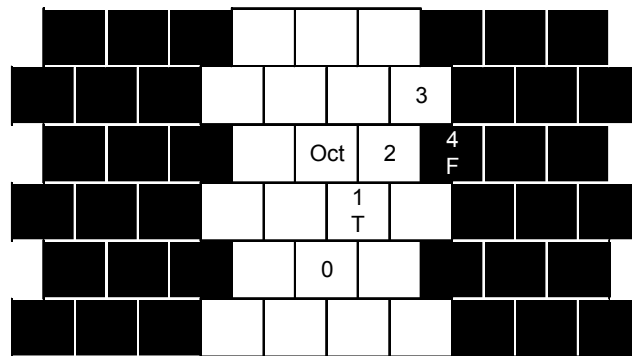


Figure 6. $F = +4T$, or $F = O^{-1} T^5$, magic.

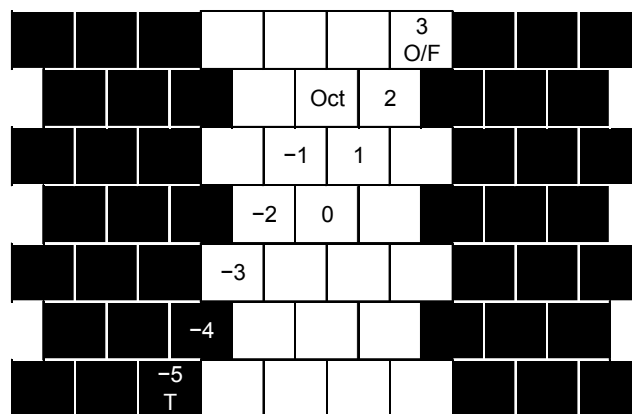


Figure 7. $T = -5/3(O/F)$, or $T = O^1(O/F)^{-5/3}$, porcupine.

For more information about the mapping of the unit of repetition and s-chain to the button-field, see Section 6.

2.3.1 Algorithmic Mapping from Temperament to Generating Set

This section is a description of an algorithm to calculate an optimal (according to the above criteria) generating set from the commas that are tempered to unison.

The primary consonances, as defined in Section o, up to a given limit, are calculated. For example, the primary consonances up to a prime limit of 11, ranked by limit and then by the $2 / (p + q)$ consonance estimate are:

2/1

3/2

4/3

5/4

5/3

6/5

8/5

$7/4$
 $7/5$
 $7/6$
 $8/7$
 $10/7$
 $12/7$

 $11/6$
 $11/7$
 $11/8$
 $11/10$
 $12/11$
 $14/11$
 $16/11$
 $20/11$.

Every possible *combination-of-primary-consonances* comprising $2/1$ and exactly one primary consonance of every other prime limit is calculated. For a prime limit of p_L , there are $\prod_{i=1}^{L-1} 2(L-i)$ possible combinations – a number that rapidly increases with the upper prime limit. For example, when $p_L = 5$, $L = 3$, so there are 8 combinations; when $p_L = 7$, $L = 4$, so there are 48; when $p_L = 11$, $L = 5$, there are 384; when $p_L = 13$, $L = 6$, there are 3,840; etc..

Given a comma sequence (Smith 2005), which consists of exactly one comma of each prime limit from p_3 up to p_L , substitution is used to derive new commas that consist of each combination of just three prime factors, one of which must be p_1 .⁵ This *derived-comma-set* consists of $\sum_{i=1}^{L-2} i$ commas. Substitution is possible because all commas in the sequence are tempered to 1 in the temperament and so can be considered to equal one another. For example, the commas in the comma sequence,

$$\begin{aligned}
 81/80 &= 2^{-4} 3^4 5^{-1} = p_1^{-4} p_2^4 p_3^{-1}, \\
 126/125 &= 2^1 3^2 5^{-3} 7^1 = p_1^1 p_2^2 p_3^{-3} p_4^1, \\
 385/384 &= 2^{-7} 3^{-1} 5^1 7^1 11^1 = p_1^{-7} p_2^{-1} p_3^1 p_4^1 p_5^1,
 \end{aligned}$$

can be substituted to produce the derived-comma-set:

$$\begin{aligned}
 81/80 &= 2^{-4} 3^4 5^{-1} = p_1^{-4} p_2^4 p_3^{-1}, \\
 (81/80)^3 (126/125)^{-1} &= 2^{-13} 3^{10} 7^{-1} = p_1^{-13} p_2^{10} p_4^{-1},
 \end{aligned}$$

⁵ It is possible to use a comma with a prime limit lower than p_3 , but the resulting temperament may be rank-1. When a rank-1 temperament is mapped to a 2-dimensional button lattice, the unused dimension can be used for any interval of choice. For instance in Table 1, the unused dimension in the $F = +^3/5O$ and $F = +^7/12O$ temperaments is algorithmically assigned to T , because this is the most familiar higher than 3-limit primary consonance. It is not, however, a precondition of tempering out these commas.

$$(81/80)^4 (126/125) (385/384)^{-1} = 2^{-24} 3^{13} 11^1 = p_1^{-24} p_2^{13} p_5^1,$$

$$(81/80)^{-1} (126/125)^2 = 2^6 5^{-5} 7^2 = p_1^6 p_3^{-5} p_4^2,$$

$$(81/80)^3 (126/125)^{-4} (385/384)^4 = 2^{-44} 5^{13} 11^4 = p_1^{-44} p_3^{13} p_5^4,$$

$$(81/80) (126/125)^3 (385/384)^{10} = 2^{-71} 7^{13} 11^{10} = p_1^{-71} p_4^{13} p_5^{10}.$$

These derived commas are then grouped into *p_g-derived-comma-sets* comprising all the commas with the prime factor *p_g*, where *g* is an integer from 2 to *L*. For example the *p₂-derived-comma-set*, taken from the above derived-comma-set is

$$2^{-4} 3^4 5^{-1} = p_1^{-4} p_2^4 p_3^{-1},$$

$$2^{-13} 3^{10} 7^{-1} = p_1^{-13} p_2^{10} p_4^{-1},$$

$$2^{24} 3^{-13} 11^{-1} = p_1^{24} p_2^{-13} p_5^{-1};$$

the *p₃-derived-comma-set* is

$$2^{-4} 3^4 5^{-1} = p_1^{-4} p_2^4 p_3^{-1},$$

$$2^6 5^{-5} 7^2 = p_1^6 p_3^{-5} p_4^2,$$

$$2^{-44} 5^{13} 11^4 = p_1^{-44} p_3^{13} p_5^4.$$

Each comma in a given *p_g-derived-comma-set* is factorised by every combination-of-primary-consonances to produce combinations of *primary-consonance-factors*.

The exponents of the primary-consonance-factors in each of these combinations are divided by the value of the exponent of the primary-consonance-factor whose limit is neither *p₁* nor *p_g* to produce *reduced-primary-consonance-factors*.

For each combination of reduced-primary-consonance-factors, the absolute value of the lowest common multiple of the exponents of the *p₁* primary-consonance-factors in all the commas in the given *p_g-derived-comma-set* is calculated. This is termed the *denominator-of-the-octave's-exponent*.

For each combination of reduced-primary-consonance-factors, the absolute value of the lowest common multiple of the exponents of the *p_g* primary-consonance-factors in all the commas in the given *p_g-derived-comma-set* is calculated. This is termed the *denominator-of-the-generator's-exponent*.

This process is repeated for all the *p_g-derived-comma-sets*.

The resulting combinations of reduced-primary-consonance-factors are then filtered accordingly:

1. The combinations whose denominator-of-the-octave's-exponent is the minimum are kept.
2. The combinations derived from the *p_g-derived-comma-set* with the lowest value of *g* are kept.
3. The combinations whose denominator-of-the-generator's-exponent is the minimum are kept.

4. The combination where the p_g -limit primary consonance has the highest $2/(p+q)$ consonance value is kept.

The exponents of the final remaining combination of reduced-primary-consonance-factors are multiplied by -1 , and this provides the redundant generator: on the right hand side of the redundant generator's equals sign is the product of the 2-limit and p_g -limit primary consonances factors with their exponents; on the left hand side is the remaining primary consonance (with no exponent).

The non redundant generators are: O with an exponent equal to that combination's reciprocal of the denominator-of-the-octave's-exponent; the p_g -limit primary consonance with an exponent equal to the reciprocal of the denominator-of-the-generator's-exponent.

2.3.2 Categorisation of Tuning Systems

The rank-2 regular tuning systems resulting from this algorithm can be broken down into the following categories:

- When the non-redundant generators are O^1 and another primary consonance (such as F^1 , $(O/F)^1$, T^1 , S^1 , $(O/S)^1$, D^1 , etc.), the temperament can be expressed by a single *s-chain tuning system*.
- When the non-redundant generators are $O^{1/n}$ (where n is an integer > 1) and a primary consonance (such as F^1 , $(O/F)^1$, T^1 , S^1 , $(O/S)^1$, D^1 , etc.), the temperament can be expressed by an “*n-tuple s-chain tuning system*” (e.g. “double” F -chain, “triple” T -chain), which consists of n *s*-chains repeated at equal divisions of the octave.
- When the non-redundant generators are O^1 and an integral root of a primary consonance (such as $F^{1/m}$, $(O/F)^{1/m}$, $T^{1/m}$, $S^{1/m}$, $(O/S)^{1/m}$, $D^{1/m}$, etc., for m is an integer > 1), the temperament can be expressed by an “*mth of an s-chain tuning system*” (e.g. a ninth of an F -chain tuning system).
- When the non-redundant generators are $O^{1/n}$ and $F^{1/m}$, or $(O/F)^{1/m}$, or $T^{1/m}$, or $S^{1/m}$, or $(O/S)^{1/m}$, or $D^{1/m}$, etc., (for m and n are integers > 1) the temperament can be expressed by an “*n-tuple mth of an s-chain tuning system*”.

It should be noted that any temperament can be hosted by more than one regular tuning system (as categorised above), but the system of choosing the optimal (i.e. simplest) generating set, as described in Section 2.3, makes this a valid means to categorise temperaments.

Conversely, any given tuning of a regular tuning system may be able to host more than one different temperament. For instance, in an F -chain tuning system where $O^{7/12} \leq F \leq 3/2$, it is reasonable to assert that both meantone and schismatic temperaments can be hosted – for more information about the impact of *s*-tunings on temperament see Section 2.4.

Table 1 shows the tuning systems produced by the above algorithm for a selection of different 5-limit temperaments. The temperaments are categorised according to the tuning system that can host them. For reference, the “common” names that have been given to the various temperaments and commas by the micro-tuning community are also included (Erlich and Lumma):

Table 1. Generating sets produced by the above algorithm for 5-limit temperaments.

Tuning system	Generators: $s(U), s(G), s(R)$:complex form	$s(R)$: simple form	Temperament: common name	Comma: prime factorisation	Comma: common name
s-chains:					
<i>F</i> -chain	$O, F, T = O^1 F^{-1}$	$T = -1F$	Father	$2^4 3^{-1} 5^{-1}$	Diatonic semitone
	$O, F, T = O^2 F^{-3}$	$T = -3F$	Mavila	$2^{-7} 3^3 5^1$	Major chroma
	$O, F, T = O^{-2} F^4$	$T = +4F$	Meantone	$2^{-4} 3^4 5^{-1}$	Syntonic comma
	$O, F, T = O^5 F^{-8}$	$T = -8F$	Schismatic	$2^{-15} 3^8 5^1$	Schisma
	$O, F, T = O^{-5} F^9$	$T = +9F$	Superpythagorean	$2^{12} 3^{-9} 5^1$	-
	$O, F, T = O^{-9} F^{16}$	$T = +16F$	-	$2^{-25} 3^{16} 5^{-1}$	-
<i>T</i> -chain	$O, T, F = O^0 T^2$	$F = +2T$	Dicot	$2^{-3} 3^{-1} 5^2$	Minor chroma
	$O, T, F = O^{-1} T^5$	$F = +5T$	Magic	$2^{-10} 3^{-1} 5^5$	Small diesis
	$O, T, F = O^{-2} T^8$	$F = +8T$	Würschmidt	$2^{17} 3^1 5^{-8}$	Würschmidt comma
	$O, T, F = O^{-3} T^{11}$	$F = +11T$	-	$2^{24} 3^1 5^{-11}$	-
	$O, T, F = O^{-4} T^{14}$	$F = +14T$	-	$2^{31} 3^1 5^{-14}$	-
<i>S</i> -chain	$O, S, F = O^2 S^{-2}$	$F = -2S$	Dicot	$2^{-3} 3^{-1} 5^2$	Minor chroma
	$O, S, F = O^{-1} S^2$	$F = +2S$	Beep	$3^3 5^{-2}$	Large limma
	$O, S, F = O^5 S^{-6}$	$F = -6S$	Hanson	$2^{-6} 3^{-5} 5^6$	Kleisma
	$O, S, F = O^{-9} S^{13}$	$F = +13S$	Parakleismic	$2^8 3^{14} 5^{-13}$	Parakleisma
<i>n</i>-tuple s-chains:					
<i>F</i> -chains: $n = 2$	$O^{1/2}, F, T = O^{3/2} F^{-2}$	$T = +^{3/2}O - 2F$	Diaschismatic	$2^{11} 3^{-4} 5^{-2}$	Diaschisma
	$n = 3$	$O^{1/3}, F, T = O^{1/3}$	Augmented	$2^7 5^3$	Great diesis
	$n = 3$	$O^{1/3}, F, T = O^{8/3} F^{-4}$	Misty	$2^{26} 3^{-12} 5^{-3}$	-
	$n = 4$	$O^{1/4}, F, T = O^{-1/4} F^1$	Diminished	$2^3 3^4 5^{-4}$	Major diesis
<i>T</i> -chains: $n = 5$	$O^{1/5}, T, F = O^{3/5}$	$F = +^{3/5}O$	Blackwood	$2^8 3^{-5}$	Limma
	$n = 12$	$O^{1/12}, T, F = O^{7/12}$	Aristoxenean	$2^{-19} 3^{12}$	Pythagorean comma
m^{th} of s-chains:					
<i>F</i> -chain: $m = 4$	$O, F^{1/4}, T = O^{-1} F^{9/4}$	$T = +^{9/4}F$	Tetracot	$2^5 3^{-9} 5^4$	Minimal diesis
<i>T</i> -chain: $m = 3$	$O, T^{1/3}, F = O T^{-4/3}$	$F = -^{4/3}T$	Negri	$2^{-14} 3^3 5^4$	-
<i>S</i> -chain: $m = 2$	$O, S^{1/2}, F = O^{-2} S^{7/2}$	$F = +^{7/2}S$	Semisixths	$2^2 3^9 5^{-7}$	-
<i>O/F</i> -chain: $m = 3$	$O, (O/F)^{1/3}, T = O^1 (O/F)^{-5/3}$	$T = -^{5/3}O/F$	Porcupine	$2^1 3^{-5} 5^3$	Maximal diesis

2.4 X_Tunings

The above table shows, for each temperament, the simplest possible tuning system, and mapping of primary consonances to that tuning system. It also indicates the simplest way for any given temperament, as a whole, to be tuned; which is to adjust the tuning of the generating interval.

2.4.1 Optimal Tunings

Every temperament will be given an *optimal tuning* (for harmonic timbres) determined using sensory dissonance criteria. The tuning that provides the lowest sensory dissonance for the octave doubled primary chords is considered to be optimal. In the unlikely circumstance that the optimal tuning differs according to whether the primary chord is major or minor, a choice will be presented to the artist.

For example, sensory dissonance criteria indicate that the optimal tuning for both the major and minor triad in a meantone temperament is $\frac{1}{4}$ -comma, i.e. $F = (3/2)^*(81/80)^{-1/4}$.

The optimal tuning forms part of an *X_Temperament preset*, which will be further explained in Section 2.7.

2.4.2 Tuning Ranges

Given a tuning system that is an *s-chain*, the most straightforward way to alter its tuning is for the tuning of *s* to be controllable – so in *F-chain* tuning systems, the ratio of *F* is changed; in *T-chain* tuning systems the ratio of *T* is changed; in *S-chain* tuning systems the ratio of *S* is changed.

It is necessary to determine the range of tunings over which all of the intervals of a temperament can actually function as effective signifiers of their just referents. Determining the tuning boundaries beyond which signification fails, however, is not only arbitrary and subjective, but also context dependent. The means by which signification can occur are complex – in any tuning that is used to realise a conventionalised musical system such as common practice tonality, the thirds and sixths are not just tuning analogues of their just referents, they also serve as functional and stylistic devices. For example in the “expressive intonation” used by classical violinists, major thirds may be played sharper even than those found in Pythagorean tuning; these thirds still signify the pure harmonic major thirds but use a more arbitrary (i.e. conventionalised) mode of relationship. Such conventionalisation allows for the tunings of tempered intervals to be more radical than might be expected from analysing harmonies isolated from these conventionalised contexts.

Furthermore, and of particular importance to this paper, the range of tunings within which an interval can be heard as signifying its just referent can be expanded through the use of similarly adjusted spectra – *X_Spectra*. It is necessary, therefore, to find an even broader means of determining sensible valid tuning ranges for any given temperament.

There are two complementary methods, which are detailed in the subsequent two sections: MOS Scale cardinality range, and Purer-Tuning range.

2.4.3 MOS:Cardinality Range

Definition: A *moment of symmetry scale (MOS scale)* is one that is constructed from a complete *s-chain*, such that the intervals between adjacent scale notes have only two sizes (Wilson).

MOS scales have constant structure, which means that every interval always spans the same number of notes (Grady); MOS scales are distributionally even (Clough, Engebretsen and Kochavi), which means that they have only two step sizes distributed as evenly as possible. These two properties make MOS scales ideal for melodic purposes, and they sound intrinsically complete and coherent – two familiar examples of MOS scales are the five note pentatonic scale and the seven note diatonic scale. When they are built from a generator that is a primary consonance, as they are in many *X_Temperaments*, they are also ideal for harmonic purposes. Being a MOS scale, however, is not a precondition for coherent tonal function (the harmonic minor scale is an example of a non-MOS scale with coherent tonal function), but it is fair to say

that a MOS scale which does exhibit functional coherence, like the diatonic scale, is to be highly valued.

Given the precise tuning of a generator, MOS scales are relatively easy to compute, and so are a convenient means to identify coherent sounding scales.

For any given tuning of the s generator, there will be a set of MOS scales of ascending cardinality. The quarter-comma meantone generator, $s = (3/2)(81/80)^{-1/4}$, gives the following set of MOS scales: (1, 2, 3), 5, 7, 12, 19, 31, 50, The Pythagorean generator, $s = 3/2$, gives the following set of MOS scales: (1, 2, 3), 5, 7, 12, 17, 29, 41, 53,

Conversely, any MOS scale of given cardinality has a *valid tuning range* within which it can be generated. These valid tuning ranges mark natural tuning boundaries for *scalic recognisability*, as will be fully explained later.

MOS scales can be quickly identified using Figure 8, which shows the valid tuning ranges for all MOS scales of cardinalities 2 to 24. The size of the generator is zero at the 12 o'clock position, 100 cents at the one o'clock position, 200 cents at the two o'clock position, and so on all the way round until the generator reaches 1200 cents (an octave) back at the 12 O'clock position. The validity ranges are shown as arc lengths terminated by radial lines which represent the generator tunings at which equal temperaments occur; MOS scale cardinalities are shown by the distance from the centre of the circle – so an MOS scale with 10 notes is shown in the tenth ring.

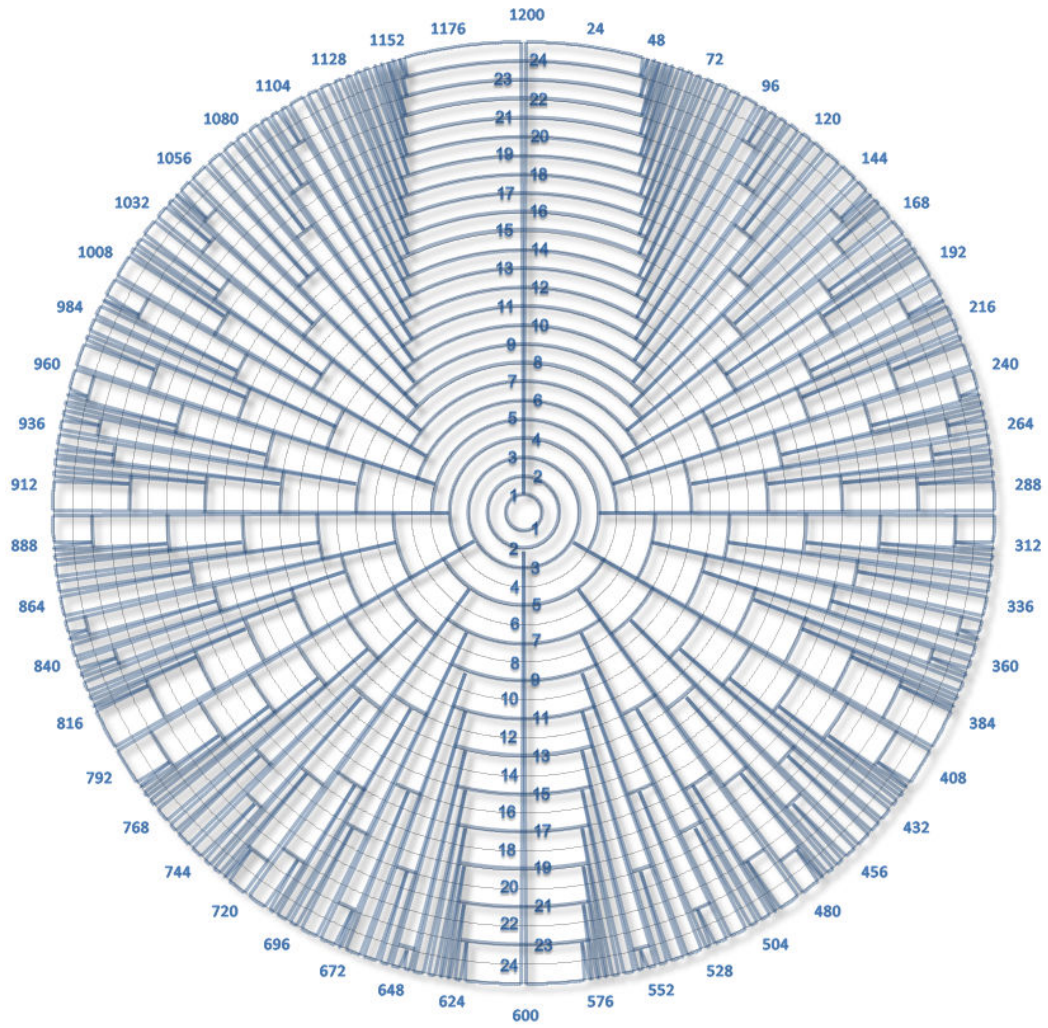


Figure 8. Valid tuning ranges for MOS scales of cardinalities 2 to 24.

The tunings at which a MOS scale of given cardinality becomes invalid or its internal structure changes, mark the natural tuning boundaries for that scale's *recognisability*.

This can be illustrated by examining the 7-note MOS scale. Starting with a generator F tuned to $O^{7/12}$, the scale consists of five large steps and two small steps, which is the familiar form of the diatonic scale.

As the F -tuning is increased, the tones get larger in size and the semitones get smaller, and when F reaches $O^{3/5}$, the semitones disappear completely; above $O^{3/5}$, there is no seven note MOS scale available (until $F = O^{2/3}$), because the seven note scale contains *three* step sizes.

As the F -tuning is decreased below $O^{7/12}$, the tones get smaller and the semitones get larger, and when F reaches $O^{4/7}$, the tones and semitones become the same size; below $O^{4/7}$, the MOS scale's internal structure changes so that it now has five small intervals and two large intervals, a scale structure which is the inverse of the familiar diatonic scale.

The tuning range of $O^{4/7} < F < O^{3/5}$ is the range of *recognisable diatonic tunings* (Blackwood), because above and below this range the familiar seven note diatonic (MOS) scale can no longer be generated.

The seven note diatonic scale has a recognisable tuning range equivalent to the valid tuning range of a MOS scale of cardinality 12 – the next cardinality above 7. This is generally true for all MOS cardinalities.

Definition: The *tuning range of recognisability* for any MOS scale of a given cardinality is equivalent to the valid tuning range of the next higher cardinality.

The valid tuning range for a MOS scale of given cardinality is designated “MOS:Cardinality” – e.g. “MOS:12” represents the valid tuning range of the MOS scale of cardinality 12 that is found at the current tuning.

Using the above chart, it is easy to see how the choice of MOS:Cardinality affects the valid tuning range. With a current F -tuning of 705 cents: setting the MOS:Cardinality to MOS:46 gives the valid tuning range of $703.45 < F < 705.88$, which is one of the ranges of recognisability for a 29 note MOS scale; dropping the MOS:Cardinality down to MOS:29 gives the tuning range of $700.00 < F < 705.88$, which is one of the ranges of recognisability for 17 note MOS scale; dropping the MOS:Cardinality down to MOS:17 gives the tuning range $700.00 < F < 720.00$, which is one of the ranges of recognisability for 12 note MOS; etc..

MOS-cardinality tuning ranges will be presented to the artist in the following way:

- An appropriate MOS:Cardinality range will be stored as part of a *temperament preset*. By default, the “appropriate” cardinality is the highest that fully incorporates the Purer-Tuning range discussed below (for instance, an appropriate cardinality range for meantone $T = +4F$ is MOS:31, an appropriate cardinality range for schismatic $T = -8F$ is MOS:43), but this is essentially an arbitrary engineering decision. When a *temperament preset* is selected, the optimal tuning along with a suitable MOS:Cardinality range will be implemented.
- Given a tuning, the artist is given a choice of alternative MOS:Cardinalities available at that tuning.
- It should be easy for the artist to slide up and down the cardinality values – in this way the changes in the cardinality range will allow the artist to “zoom” in or out of a certain tuning area, while at all times the boundaries of the tuning range are meaningful.
- A gesture will be made available to the artist that allows for the tuning to be pushed beyond the currently selected range and into the adjacent cardinality tuning range.

2.4.4 Purer-Tuning Range

5-limit or higher just intonations are the *only* regular tuning systems that can provide justly tuned perfect fifths, major thirds, major sixths, and all of their octave inversions. A rank-2 regular temperament can only have one of these primary consonances purely tuned at a time. For instance, in the meantone temperament:

- when $F = (3/2) (81/80)^{-1/3}$ (\flat -comma), S is purely tuned ($S = 5/3$) while F and T are not;

- when $F = (3/2)(81/80)^{-1/4}$ ($1/4$ -comma), T is purely tuned ($T = 5/4$) while F and S are not;
- when $F = 3/2$ (o-comma), F is purely tuned while T and S are not.

Figure 9 illustrates how $1/4$ -comma and o-comma mark meaningful tuning boundaries. From o-comma to $1/4$ -comma, the tuning of T and S improve. From $1/4$ -comma to $1/2$ -comma, the tuning of S improves. Below $1/2$ -comma, F , T and S only get worse; above o-comma, F , T and S only get worse. It is only within the $1/2$ -comma $\leq F \leq$ o-comma range that *Purer-Tunings* are available.

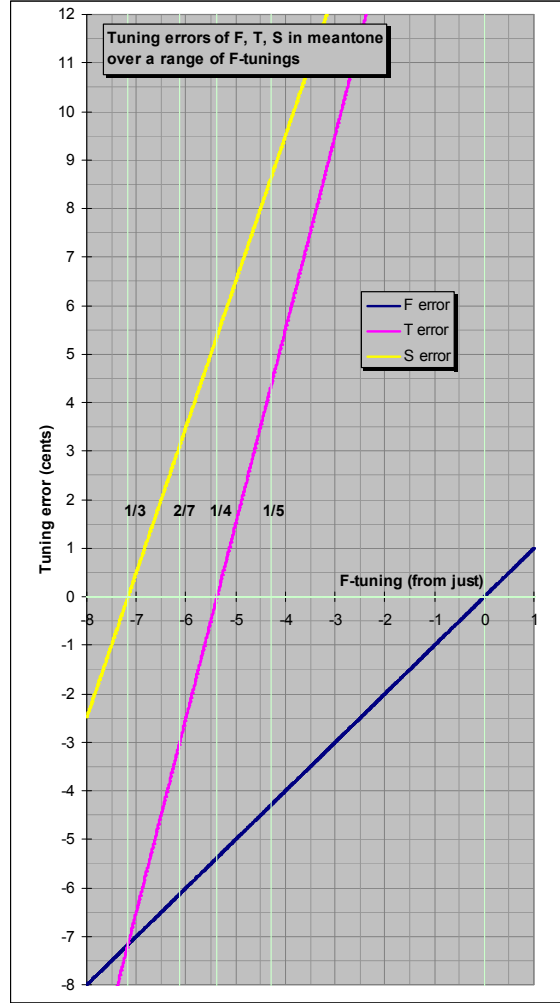


Figure 9. Tuning errors in meantone temperament.

A similar process can be applied to all possible temperaments by finding the s -tunings that provide pure F , T , and S , and setting the temperament's Purer-Tuning range to extend from the minimum to maximum of those s -tuning values.

For instance, in the schismatic temperament $T = O^5/F^8$, so $F = (O^5/T)^{1/8}$; $F = OT/S$ (by definition), so $F = (O^6/S)^{1/9}$. This means that:

- when $T = 5/4$, $F = 701.711$ cents;

- when $S = 5/3$, $F = 701.738$ cents;
- when $F = 3/2$, $F = 701.955$ cents;

so the Purer-Tuning range is $701.711 \leq F \leq 701.955$ cents.

For 7-limit or higher temperaments, it is necessary to check the tuning of the higher limit primary consonances as well. The Purer-Tuning range is always set by the minimum and maximum tunings that provide a purely tuned primary consonance at the designated limit.

For instance, the 7-limit meantone temperament with the redundant generators $T = +4F$ and $D = +10F$ has the following tunings that provide just primary consonances:

- when $F = 3/2$, $F = 701.955$ cents;
- when $T = 5/4$, $F = 696.578$ cents;
- when $S = 5/3$, $F = 694.786$ cents;
- when $D = 7/4$, $F = 696.882$ cents;
- when $D/T = 7/5$, $F = 697.085$ cents;
- when $D/F = 7/6$, $F = 696.319$ cents;

so the Purer-Tuning range is $696.319 \leq F \leq 701.955$ cents.

2.5 X_Temperaments

Definition: An *X_Temperament* is a rank-2 p -limit regular temperament that tempers out a designated p -limit comma sequence, by mapping a p -limit just intonation to an optimal rank-2 *X_Generating* set, and has a valid tuning range as determined by Purer-Tuning or an arbitrary MOS-cardinality range.

Note that this definitional system takes no account of whether or not the tuning range is psycho-acoustically valid – that is for the artist to determine, not the theorist; the artist can choose her preferred cardinality range for any given temperament – going up in cardinality to give finer granularity to explore a certain tuning region, going down in cardinality to explore new areas, or simply limiting herself to the range of Purer-Tunings.

An *X_Temperament* is named using the following unambiguous terminology:

Redundant generator(s) | valid tuning range.

The redundant generator is preferably expressed in its simple form.

When using unusual temperaments, remembering how the primary consonances are derived from the generator is one of the hardest things to do. The beauty of the above naming scheme is that the temperament is named after its redundant generator, which immediately informs the artist, in the simplest possible terms, how the primary consonances are generated (i.e. mapped to the tuning system). The key to unlocking the temperament is in its name, and no longer needs to be looked up separately.

When possible, the name of the temperament should be displayed so that it can be easily referenced by the artist whenever needed.

When using MOS to determine the valid tuning range, the designation given to the X_Temperament is, therefore:

Redundant generator(s) | MOS:Cardinality.

So a meantone temperament expressed over the “historical” range of $O^{11/19} \leq F \leq O^{7/12}$ is designated:

$T = +4F$ | MOS:31.

A “meantone” temperament expressed over the wider range of diatonic recognisability, $O^{4/7} \leq F \leq O^{3/5}$, is designated:

$T = +4F$ | MOS:7.

A schismatic temperament expressed over the range $O^{17/29} \leq F \leq O^{7/12}$ is designated:

$T = -8F$ | MOS:41.

When using Purer-Tuning to determine the valid tuning range, the designation given to the X_Temperament is:

Redundant generator(s) | Purer-Tuning prime limit.

So a 5-limit meantone temperament expressed over the 1/3-comma to 0-comma range is designated:

$T = +4F$ | PT:5.

A 7-limit meantone expressed over the 696.32 to 701.96 cents Purer-Tuning range is designated:

$T = +4F, D = +10F$ | PT:7.

When used more generally, the tuning range can be omitted, so the meantone temperament can be simply called $T = +4F$, and the “magic” temperament can be called $F = +5T$.

2.6 Mapping Tuning Ranges to Controllers

The s-tuning can be controlled at three levels of granularity:

- only once at the start of a given piece (granularity: per piece)
- a few times over the course of a piece, a la key changes (granularity: per section of a piece)
- frequently, a la pitch bending or other note-level expressive effects (granularity: per phrase or per note).

If the tuning changes are to be “per piece”, then a software-based UI is sufficient. If they are per section, then the tuning can be loaded as part of a pre-set (or a user-defined variation thereon). If the tuning can change as an expressive effect at the granularity of a phrase or a note, though, then the s-tuning needs to be associated with a joystick, wheel, or lever control mechanism.

The control mechanism that controls the tuning will be assumed to have three significant points of travel – the *lower endpoint*, the *centre point*, the *upper endpoint*. These three points are mapped to significant tunings within the tuning range.

Using the MOS:Cardinality range, the centre point corresponds to the tuning which defines that cardinality, and the lower and upper endpoints correspond to the limits of the valid tuning range for that cardinality.

For example:

- in $T = +4F \mid \text{MOS:}31$, the lower endpoint is 19-tet, the centre point is 31-tet, the upper endpoint is 12-tet;
- in $T = +4F \mid \text{MOS:}12$, the lower endpoint is 7-tet, the centre point is 12-tet, the upper endpoint is 5-tet.

In Purer-Tuning, the centre point corresponds to the most centrally located tuning that gives a purely tuned consonance; the lower and upper end points will correspond to the tunings which provide the pure tunings for the remaining two consonances.

For example:

- in $T = +4F \mid \text{PT:}5$, the lower endpoint will correspond to $F = (3/2) (81/80)^{-1/3}$, the centre point will correspond to $F = (3/2) / (81/80)^{-1/4}$, the upper endpoint will correspond to $F = 3/2$;
- in $T = -8F \mid \text{PT:}5$, the lower endpoint will correspond to $F = (3/2) * 1/8$ -schisma, the centre point to $F = (3/2) * 1/9$ -schisma, the upper endpoint will correspond to $F = 3/2$;
- in higher-limit Purer-Tunings, the centre point will correspond to the most central tuning that provides a pure consonance.

The function relating the distance moved by the controller to the change in s -tuning should be linear to $\log(s)$, because pitch change is perceived logarithmically. The three-point mappings described above mean that the rate of pitch change, relative to controller travel, above and below the centre point will be different. Alternatively, a non-linear interpolation function that provides a gradual gradient change, but which still runs through the three points, can be used.

2.7 X_Temperament Presets

An X_Temperament preset contains the following data and when it is selected by the artist it is applied in the following manner. The X_Temperament preset will function differently according to whether the artist prefers to work with MOS:Cardinalities or Purer-Tunings:

- Temperament name – using the simple form of the redundant generator, and a range defined by MOS:Cardinality or Purer-Tuning limit – as described in Section 2.5. This information should be clearly displayed to the artist on any relevant display device, because not only does it define the X_Temperament, it also provides useful information about its structure and range.
- Recommended s -tuning range – when the controller is mapped to s -tuning, the centre and end-points are set to tunings defined by the MOS:Cardinality or Purer-Tuning Range, as described in Section 2.6.

- Optimal tuning – when the controller is not mapped to *s*-tuning, the tuning is set to the optimal tuning as described in Section 2.4.1.
- Specialist mappings – when specialist *X_Layouts* are required, they are automatically applied as according to Section 6.3.1.

2.8 Centring and Re-centring

When the tuning of the *s*-generator is altered, the tuning changes must occur in reference to one particular note within the *s*-chain; so when the *s*-tuning is increased, notes in the chain above the centre note will go up in pitch, notes in the chain below the centre note will go down.

By default, that centre note will be Re, however the artist will be given the option to choose Do or La as alternative centre notes. Re is the central note in the chain of fifths making up the MOS:7 diatonic scale, and so provides a natural centre for tuning changes; Do ensures that the pitch of the tonic of major tonality never changes with *s*-tuning alterations; La ensures that the pitch of the tonic of minor tonality never changes with the *s*-tuning.

Re-centring is the name given to the ability to designate different pitches to the centre-note of the *s*-chain; this is achieved with a gesture that shifts the notes up or down by one step of the *s*-chain. In temperaments using conventional *F*-chain tuning systems, this amounts to transposing up and down by perfect fifths; so at a simpler level, the Re-centring can be understood as a simple transpose control. Re-centring can be used when changing key, and it ensures that the full range of remote intervals is always available – whatever the key being used. This is particularly important when using less efficient temperaments such as $T = -8F$ which use up a large area of the *s*-chain, otherwise only a limited number of different keys are actually playable.

2.9 Equal Temperaments

The passage through a tuning range, takes us through a number of *n*-tets. For instance, in the range that produces recognisably diatonic MOS scales, $O^{4/7} < F < O^{3/5}$, the tuning passes through the following equal temperaments (where $n \leq 53$): 7, 47, 40, 33, 26, 19, 50, 31, 43, 12, 53, 41, 29, 46, 17, 39, 22, 49, 27, 32, 37, 42, 47, 52, 5.

Within any temperament there is no need to learn and relearn fingering patterns for each new *n*-tet. They are always the same, because they are just points on the 2-dimensional *s*-tuning continuum.

For instance using the $T = +4F$ X_Temperament, intervals are mapped to buttons in the following way:

Gb4		Ab4		Bb4		C5		D5		E5		F#5		G#5		A#5			
dim. seventh		dim. octave		Eb4		F4		G4		A4		B4		C#5		D#5		E#5	
dim. fourth		dim. fifth		minor sixth		minor seventh		octave		E4		F#4		G#4		A#4			
Cb3		Db3		minor second		minor third		perfect fourth		perfect fifth		major sixth		major seventh		D#4		E#4	
Gb2		Ab2		Bb2		C3		unison		major second		major third		aug. fourth		aug. fifth			
Cb2		Db2		Eb2		F2		G2		A2		B2		C#3		aug. unison		aug. second	

Figure 10. Default mapping of sub-octave intervals to the Thummer when using the $T = +4F$ X_Temperament.

As the tuning is changed from one equal temperament to another, the correct step numbers of that n -tet automatically line up to the correct point – so that given an unchanging temperament, the position of the notes signifying jO, jF, jT, jS etc. never changes (assuming the tuning stays within a reasonable tuning range). The following figures show the step number locations for n -tets with $n \leq 53$ in the MOS:31 tuning range, as the s -tuning is progressively raised:

19-tet									
Gb4	Ab4	Bb4	C5	D5	E5	F#5	G#5	A#5	
15 Cb4	18 Db4	Eb4	F4	G4	A4	B4	C#5	D#5	E#5
7 Gb3	10 Ab3	13 Bb3	16 C4	19 D4	E4	F#4	G#4	A#4	
Cb3	Db3	2 Eb3	5 F3	8 G3	11 A3	14 B3	17 C#4	D#4	E#4
Gb2	Ab2	Bb2	C3	0 D3	3 E3	6 F#3	9 G#3	12 A#3	
Cb2	Db2	Eb2	F2	G2	A2	B2	C#3	1 D#2	4 E#2

Figure 11. Step locations for 19-tone equal temperament.

50-tet

	Gb4	Ab4	Bb4	C5	D5	E5	F#5	G#5	A#5	
29	37									
Cb4	Db4	Eb4	F4	G4	A4	B4	C#5	D#5	E#5	
	18	26	34	42	50					
	Gb3	Ab3	Bb3	C4	D4	E4	F#4	G#4	A#4	
			5	13	21	29	37	45		
Cb3	Db3	Eb3	F3	G3	A3	B3	C#4	D#4	E#4	
					0	8	16	24	32	
	Gb2	Ab2	Bb2	C3	D3	E3	F#3	G#3	A#3	
Cb2	Db2	Eb2	F2	G2	A2	B2	C#3	3	11	
								D#2	E#2	

Figure 12. Step locations for 50-tone equal temperament.

31-tet

	Gb4	Ab4	Bb4	C5	D5	E5	F#5	G#5	A#5	
24	29									
Cb4	Db4	Eb4	F4	G4	A4	B4	C#5	D#5	E#5	
	11	16	21	26	31					
	Gb3	Ab3	Bb3	C4	D4	E4	F#4	G#4	A#4	
			3	8	13	18	23	28		
Cb3	Db3	Eb3	F3	G3	A3	B3	C#4	D#4	E#4	
					0	5	10	15	20	
	Gb2	Ab2	Bb2	C3	D3	E3	F#3	G#3	A#3	
Cb2	Db2	Eb2	F2	G2	A2	B2	C#3	2	7	
								D#2	E#2	

Figure 13. Step locations for 31-tone equal temperament.

43-tet

	Gb4	Ab4	Bb4	C5	D5	E5	F#5	G#5	A#5	
33	40									
Cb4	Db4	Eb4	F4	G4	A4	B4	C#5	D#5	E#5	
	15	22	29	36	43					
	Gb3	Ab3	Bb3	C4	D4	E4	F#4	G#4	A#4	
			4	11	18	25	32	39		
Cb3	Db3	Eb3	F3	G3	A3	B3	C#4	D#4	E#4	
					0	7	14	21	28	
	Gb2	Ab2	Bb2	C3	D3	E3	F#3	G#3	A#3	
Cb2	Db2	Eb2	F2	G2	A2	B2	C#3	3	10	
								D#2	E#2	

Figure 14. Step locations for 43-tone equal temperament.

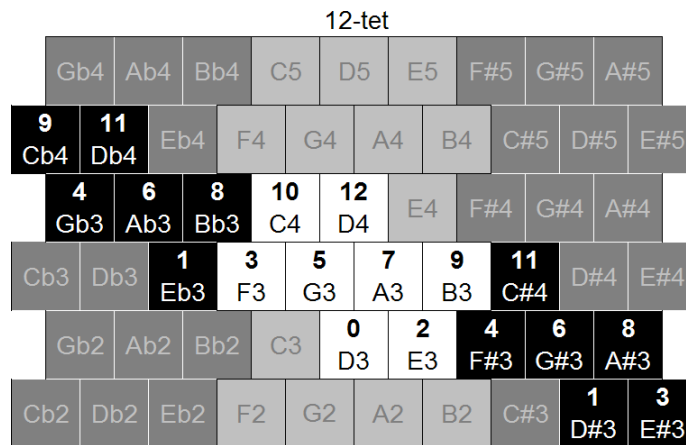


Figure 15. Step locations for 12-tone equal temperament.

2.10 Metrics for the Tuning Range

An ideal metrical system is able to express significant measurements with simple (e.g. integral) values. For example, Celsius marks the freezing point of water at 0 °C and the boiling point at 100 °C.

Within a tuning range, there are two different types of psycho-acoustically significant landmarks. The two types are incommensurable, so two different metrics are required.

2.10.1 Commatic Landmarks

Those tunings that provide a just tuning of F , T or S , are easily recognisable. The tunings where the mistunings of two of these intervals are equal have also been recognised by theorists as equable compromises. This is illustrated in Figure 9 above, where $1/3$ -comma gives a just S ; $2/7$ -comma gives equally mistuned S and T ; $1/4$ -comma gives a just T ; $1/5$ -comma gives equally mistuned T and F ; 0-comma gives a just F .

Almost all temperaments have these five significant tuning points. The standard metric, with an historical pedigree, is to describe $T = +4F$ (i.e. meantone) tunings according to the fraction of a syntonic comma that their fifth has been tempered, e.g. the $1/3$ -comma, $2/7$ -comma, values used in the previous paragraph. Extending this approach to other temperaments, and substituting their defining comma for the syntonic, and adding a + or - to indicate the direction of temperament applied to the generator, might seem like a sensible way to proceed, but it has the drawback that the same fraction will have quite different meanings in different temperaments. There is a way, however, to normalise these values so they remain consistent across temperaments: this is done by multiplying the fraction-of-comma value by the exponent of the generator required to hit the primary consonance that requires the lowest power of the generating interval. For instance, in $T = -8F$, it is possible quickly to derive that $S = -9F$ (because by definition $S = T - F$, using the simple logarithmic form and ignoring octaves), so the fraction of a comma by which the fifth is tempered must be multiplied by 9, to get the correct measurement value. This method will always give a metric where 0 means the generator is purely tuned, and 1 means that the primary consonance that requires the greatest tempering of the generator is purely tuned.

The symbol used for this metric is k , for kernel (which is what the comma is). Note that this $0 \leq k \leq 1$ range is identical to the Purer-Tuning range discussed above, and so provides the most suitable metric when using that type of X_Tuning range.

2.10.2 n -tet Landmarks

For instruments with N buttons per octave, the s -tunings that provide n -tets where $n \leq N$, mark the points where the tuning of different buttons on the keyboard become identical – these, therefore, mark easily verifiable (for the user) tuning points. Although not providing acoustically significant landmarks, some $n > N$ tunings have historical significance (e.g. meantone 31-tet, or schismatic 53-tet) and these can be included too.

For the diatonic scale, Blackwood's "R" provides a good metric for comparing meantone n -tets (Blackwood). R is calculated by dividing log values for the tone by the semitone, so that $R = 1$ corresponds to $F = O^{4/7}$, $R = \inf$ corresponds to $F = O^{3/5}$. R can be generalised for all MOS scales where, given the cardinality of that MOS scale, it is the ratio between the log sizes of the scale's large and small intervals.

The symbol used for this metric is r . Note that this means that the $1 \leq R \leq \inf$ range can, if desired, always correspond to the chosen MOS-cardinality range. It may be desirable, however, to keep the cardinality range low for this metric – either way, it provides the most suitable metric to be used in conjunction with the MOS cardinality tuning range method.

Alternatively, n -tets can also be indicated by simply using n to describe the temperament. The unit here will be t , for tone.

Alternatively, the $+$ or $-$ cent value by which the generator is tempered can also be used; cents will be abbreviated to c . It is assumed that the \pm cents value is referring to the generating interval s ; if it isn't, or extra clarification is required, the cent value can be preceded with O , F , T or S ; e.g. $F = -1.955c$, $T = +13.686c$, $S = +15.641c$.

2.10.3 Designating X_Tunings

To designate a particular temperament and tuning, the same format is used as when designating a temperament and valid tuning range as described in Section 2.5, but with the specific tuning replacing the MOS:Cardinality or Purer-Tuning range.

$T = +4F$ (meantone) temperament using a $1/4$ -comma tuning can be designated in the following ways:

- $T = +4F \mid -3/4k$
- $T = +4F \mid -5.38c$

$T = +4F$ tuned to 31-tet (using a MOS:7 metric) can be designated in the following ways:

- $T = +4F \mid 5/3r$
- $T = +4F \mid 31t$
- $T = +4F \mid -5.18c$

53-tet used to host a meantone temperament:

- $T = +4F \mid 53t$

53-tet used to host a schismatic temperament:

- $T = -8F \mid 53t$

2.11 Why Retune?

Re-tuning can be used for:

- Historically accurate performance of the existing repertoire – Pythagorean tuning for medieval music; $\frac{1}{4}$ -comma and $\frac{2}{7}$ -commas for classical music dating from the 16th to 17th centuries; 12-tet for post 18th century romantic, modern and atonal.
- Accurate performance of world music: 5-tet for Indonesian Slendro; Pythagorean for Arabic; etc..
- New and unusual tunings for new and as yet unheard musics.
- Expressive intonation in conventional tonal music – it is common performance practice for string players to move towards, and even beyond, Pythagorean tuning to give melodic passages more expressive power. This is easily achieved by pushing the tuning controller to the top of the $T = +4F \mid PT:5$ tuning range when extra emotive power is required. Because the tuning changes function polyphonically, the artist can also raise the tuning of dominant chords towards Pythagorean to increase their dissonance, and lower the tuning of tonic chords towards $\frac{1}{4}$ -comma to enhance their stability. The ability to use USB connections so that a group of Thummers can have their tuning controlled in a consistent way by a single “tuning conductor” allows for instrumentalists to think in terms of dissonance and expressiveness, without having to worry about how their own tuning must relate to the tuning of the other players.

3 X_Spectrum

A tuning is more useful when its intervals allow a large range of dissonances and consonances. With consonant intervals, it is possible to create chords that are restful and peaceful. With dissonant intervals, it is possible to create chords that imply motion and demand resolution. Together, dissonances and consonances are one of the engines that drive tonal music.

Modern psychoacoustics (Plomp and Levelt) and (Roederer) acknowledges that the consonance and dissonance of a sound are dependent not only on the intervals but also on the spectrum or timbre of the sound. Recent work (Sethares) shows how the *sensory dissonance* can be used to predict the amount of dissonance caused by a sound at a given interval with a given spectrum. Moreover, by changing either the tuning (the interval) or the timbre (the spectrum) it is possible to increase or decrease the perceived dissonance.

X_Spectrum shows how this can be accomplished in an organized way for any tuning of any X_Temperament. In the jargon of Sethares (Sethares), this provides gives a spectrum that is “related” to a given X_Temperament. Playing sounds generated with the related X_Spectrum allows the consonant intervals of the X_Temperament to be as consonant as possible, while allowing the dissonant intervals to remain dissonant.

3.1 Sensory Dissonance or Roughness

The creation of X_Spectra for particular tunings requires a means to determine the relative consonance of intervals played in different tunings and with different spectra. The *sensory dissonance* algorithm provides a means to achieve this task, and this is outlined in this section.

The idea that the beating of sine wave partials is related to the dissonance of a sound was first introduced by Helmholtz (1954), and Plomp and Levelt (1965) quantified this experimentally and related the width of the dissonance curve (the range of frequencies over which the roughness occurs) to the width of the critical band, thus providing a physically plausible mechanism. The Plomp and Levelt model for the sensory dissonance between two sinusoids with frequencies f_1 and f_2 can be conveniently parameterized (Sethares 1993) by an equation of the form

$$d(f_1, f_2, l_1, l_2) = \min(l_1, l_2) e^{-b_1 |f_1 - f_2|} - e^{-b_2 |f_1 - f_2|} \quad (1)$$

where l_1 and l_2 represent the loudnesses of the two sinusoids and where the exponents $b_1 = 3.5$ and $b_2 = 5.75$ specify the rates at which the function rises and falls.

A typical plot of this function $d()$ is shown in Figure 16.

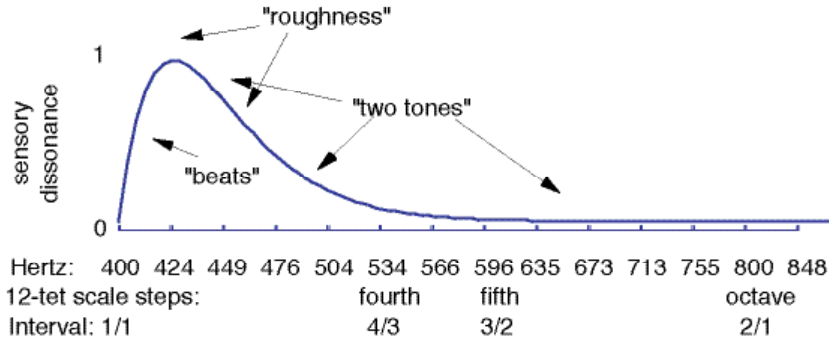


Figure 16. Normalised sensory dissonance of two simultaneously sounding sinusoids.

At first glance it is not obvious why this curve has any relevance to traditional uses of the word “dissonance” (Tenney 1988). The key is that most musical sounds do not consist of pure sine waves (such as were used in the empirical tests from which the curve was derived). Rather, they consist of a fundamental and numerous overtones that form a harmonic structure. When a sound contains many partials, the total roughness is the sum of all the $d()$'s over all pairs of partials

$$D = 0.5 \sum_i \sum_j d(f_i, f_j, l_i, l_j). \quad (2)$$

If the sound happens to be harmonic with fundamental g , then the i^{th} partial is $i \cdot g$. For such a sound, a typical dissonance curve (a plot of D over all intervals of interest) looks like Figure 17. Observe that the minima of the curve occur at or near musically sensible locations: the fifths, thirds, and sixths of familiar usage. In fact, they occur at exactly the locations of the justly intoned scales.

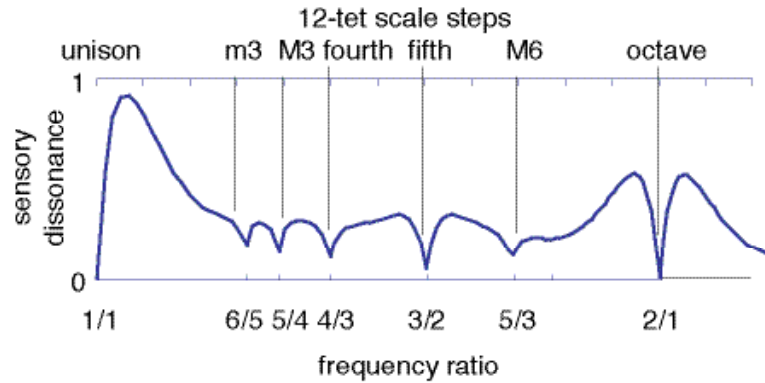


Figure 17. Dissonance curve for a spectrum with harmonic partials.

The general contour of the dissonance curve mimics common musical intuitions regarding consonance and dissonance, though there are some differences. Perhaps the most striking of these is that the dissonance curves are a function of the timbre (or the spectrum) of the sound, rather than an intrinsic property of the interval itself. This feature (and others) are discussed at length in Sethares (2004).

3.2 Tempering Harmonics

In the same way that just intonation is related to the harmonic spectra (through the process of generating a dissonance curve with minima that lie at the desired scale steps), so the tempered intonations (such as those of the X_Temperaments) can be related to spectra with tempered partials. The harmonics of a sound can be matched to the X_Temperaments in a straightforward way. Using the generating set of the tuning system results in *tempered harmonics* that can be used to create sounds with related spectra that are called X_Spectra. As will be shown, the dissonance curves generated by these X_Spectral sounds have their minima at the locations of the primary consonances of the related temperaments.

Table 2 indicates the mapping from a harmonic sound to an X_Spectral sound for the first sixteen harmonics. The first column shows the prime factorisation for each harmonic. The second column shows how these partials are derived from the 5-limit just intonation generators $jO = 3/2$, $jF = 5/4$ and $jT = 5/3$. By definition, $jF = jO^1 jT^1 jS^{-1}$, and $jT = jO^{-1} jF jS$, allowing the derivation of the partials to be expressed in terms of O , F , T ; or O , F , S ; or O , T , S .

Table 2. Factorisations of harmonic and tempered partials.

	2, 3, 5	jO, jF, jT	O, F, T	O, F, S
1	1 ¹	1	1	1
2	2 ¹	jO ¹	O ¹	O ¹
3	3 ¹	jO ¹ jF ¹	O ¹ F ¹	O ¹ F ¹
4	2 ²	jO ²	O ²	O ²
5	5 ¹	jO ² jT ¹	O ² T ¹	O ¹ F ¹ S ¹
6	2 ¹ 3 ¹	jO ² jF ¹	O ² F ¹	O ² F ¹
7	7 ¹			
8	2 ³	jO ³	O ³	O ³
9	3 ²	jO ² jF ²	O ² F ²	O ² F ²
10	2 ¹ 5 ¹	jO ³ jT ¹	O ³ T ¹	O ² F ¹ S ¹
11	11 ¹			
12	2 ² 3 ¹	jO ³ jF ¹	O ³ F ¹	O ³ F ¹
13	13 ¹			
14	2 ¹ 7 ¹			
15	3 ¹ 5 ¹	jO ³ jF ¹ jT ¹	O ³ F ¹ T ¹	O ² F ² S ¹
16	2 ⁴	jO ⁴	O ⁴	O ⁴

The tunings of the X_Spectral tempered harmonics required for any given X_Temperament tuning can be determined by choosing the correct substitution for F , T or S , as defined by its redundant generator (in its complex form). For instance, in the $T = +4F$ X_Temperament, $s(R) = T = F^4/O^2$; so to calculate the tuning of the partials go to the O , F , T column and substitute F^4/O^2 for T . This allows for all of the tunings of the 5-limit tempered harmonics to be calculated from any given F -tuning.

A similar process allows for higher-than-5-limit harmonics to be retuned to match any given higher-than-5-limit temperament. Figure 18 shows how the X_Spectral tempered harmonics of the lowest note of C on an extended Thummer button-field are related to

the other notes, when using a $T = +4F$, $D = +10F$, $L = +6F$, $H = -4F$, $V = -5F$, $N = -3F$ X_Temperament (which is suitable for tunings where F is flattened by about 5 cents). Where a note and a harmonic coincide they are bolded; prime harmonics are coloured and all octave expansions and reductions of that harmonic have the same colour; non-prime harmonics are shown in grey. Note that the position of the non-prime harmonics is determined by the position of the prime harmonics – for example, the location of the 15th harmonic is determined by adding together the locations of the 3rd and 5th harmonics (3 and 5 being the prime factors of 15).

gb	26 ab	bb	32 c	36 d	40 e	44 f#	50 g#	56 a#
cb	17 db	19 eb	f	24 g	27 a	30 b	33 c#	42 e#
gb	13 ab	bb	16 c	18 d	20 e	22 f#	25 g#	28 a#
cb	db	eb	f	12 g	a	15 b	c#	d# e# 21
gb	ab	bb	8 c	9 d	10 e	11 f#	g#	14 a#
cb	db	eb	f	6 g	a	b	c#	d# e#
gb	ab	bb	4 c	d	5 e	f#	g#	7 a#
cb	db	eb	f	3 g	a	b	c#	d# e#
gb	ab	bb	2 c	d	e	f#	g#	a#
cb	db	eb	f	g	a	b	c#	d# e#
gb	ab	bb	1 c	d	e	f#	g#	a#
cb	db	eb	f	g	a	b	c#	d# e#

Figure 18. Correspondence of notes and harmonics of a low C when using a $T = +4F$, $D = +10F$, $L = +6F$, $H = -4F$, $V = -5F$, $N = -3F$ X_Temperament.

Using Table 2, it is possible to draw the dissonance curves for selected timbres and to study the correlation of the minima of the resulting dissonance curves. For example, consider the $T = +4F$ | MOS:12 X_Temperament, which has fifths that occupy a tuning range of $O^{4/7} \leq F \leq O^{3/5}$. At one extreme, $F \leq O^{3/5}$ and the tuning is 5-tone equal temperament (5-tet). The dissonance curve is:

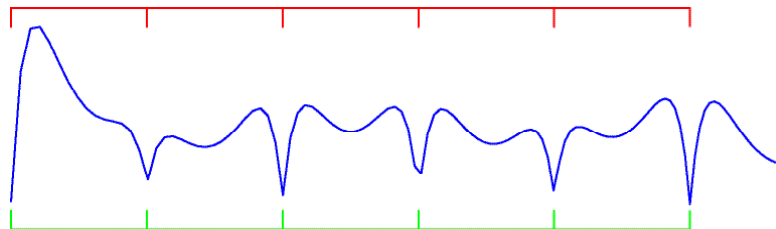


Figure 19. Dissonance curve for a tempered spectrum related to $T = +4F$ | 5-tet.

The blue curve is the sensory dissonance from unison at the left to slightly above an octave on the right. The top (red) markers indicate the scale steps of 5-tet, which are

equally spaced between the unison and the octave. The lower (green) markers indicate the locations of the primary consonances: the unison, fifth (and its inverse the fourth), the major third (and its inverse the minor sixth) and the sixth (and its inverse the minor third). In 5-tet, several of these primary consonances are degenerate. Observe the coincidence of the minima of the dissonance curve, the scale steps of the tuning, and the locations of the primary consonances.

Another $T = +4F \mid \text{MOS:12}$ tuning is 53-tet, which has 53 tones per octave. Using the same procedure as above to build 53-tet X-Spectra results in the following dissonance curve:

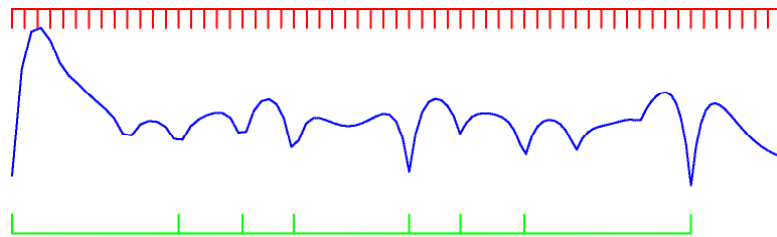


Figure 20. Dissonance curve for a tempered spectrum related to $T = +4F \mid 53\text{-tet}$.

Again, the dissonance of intervals is displayed in blue, the scale steps in red, and the primary consonances in green. Observe the coincidence between the minima of the curve and the consonances: each consonance lies at a minimum. There are also minima at other locations, and these again occur at scale steps. Compare these two figures to X-Spectra for the familiar 12-tone equal temperament (12-tet):

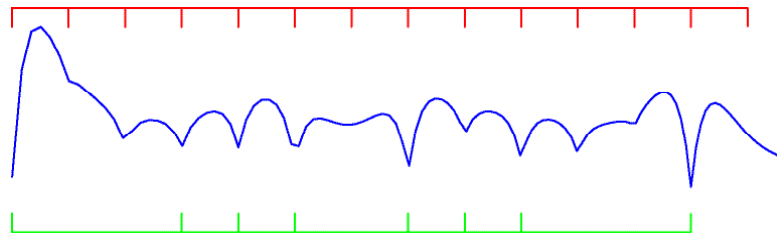


Figure 21. Dissonance curve for a tempered spectrum related to $T = +4F \mid 12\text{-tet}$.

While somewhat different in detail from the harmonic dissonance curve of the previous section (the differences are because the spectrum in the previous section used only 8 harmonic partials, whereas this one uses the 16 partials of the above table) this again displays the importance of matching the spectra and the tuning. All of the primary consonances occur at scale steps and all occur at minima of the dissonance curve.

By changing the tuning in a structured way, it is possible to retain the same fingerings on the hexagonal keyboard through a wide range of tunings. By changing the spectra (by tempering the harmonics of the sound) in an analogous way, it is possible to have these same (signified) intervals, with the same fingerings, also have similar consonance. This unifies all the tunings in a family so that a musician need only learn one set of fingerings: the consonances and dissonances can then remain similar throughout the whole tuning range.

For example, the $T = +4F \mid \text{MOS:12}$ tuning range also contains many other equal temperaments. Here are the analogous plots for 26-tet, 31-tet and 7-tet:

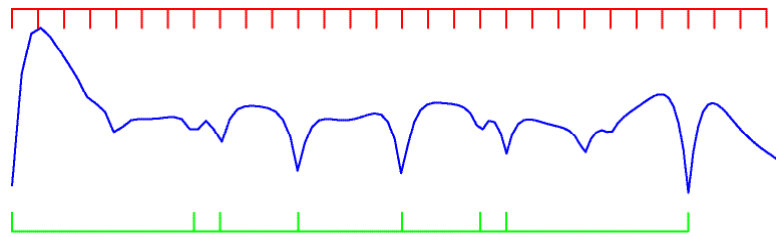


Figure 22. Dissonance curve for a tempered spectrum related to $T = +4F \mid 26\text{-tet}$.

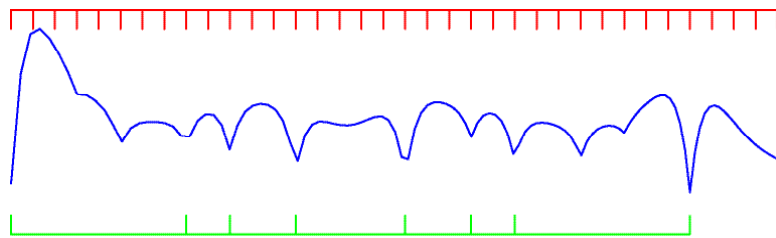


Figure 23. Dissonance curve for a tempered spectrum related to $T = +4F \mid 31\text{-tet}$.

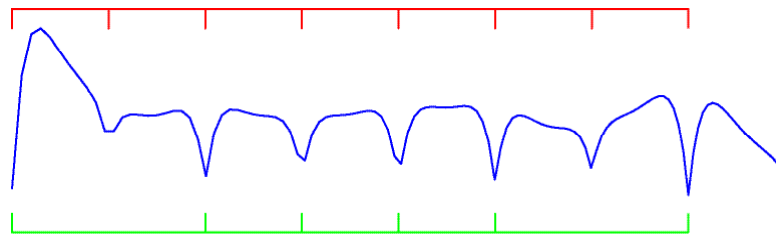


Figure 24. Dissonance curve for a tempered spectrum related to $T = +4F \mid 7\text{-tet}$.

3.3 Sensory Consonance versus Coherence

Tempering the harmonics of a tone so that they match the temperament being used, means that the sensory consonance of the prime chords, and the intervals they comprise, is maximised.

When the harmonic tempering is dynamically altered to match the tuning, chords will always sound as purely tuned as their just intonation counterparts – whatever the actual tuning chosen. This allows for temperaments to be extended beyond their traditional tuning ranges, and gives further justification for the use of the broader tuning ranges that can be provided by MOS:Cardinalities.

In this way, commas can be removed from a musical system without having to pay the “consonance penalty”. For instance, in the performance of common-practice music it is desirable to temper out the syntonic comma to avoid the commatic shift (or drift) that otherwise occurs in simple cadential chord progressions. The temperament that tempers out the syntonic comma is the familiar $T = +4F$ (meantone), but this temperament causes triads to be slightly mistuned. By using X_Spectra, the dissonance caused by this mistuning is completely eliminated, but the syntonic comma remains tempered out of the system.

The potential disadvantage is that the more a tone is spectrally tempered, the more it tends to break apart and lose its identity as a single object of perception. With extreme retuning, the tone may lose *coherence* and take on an uncertain, or multiple pitched, bell-like quality. This can be an aesthetically attractive effect, but it may not be suitable for the performance of some types of music.

There will be a parameter made available that controls the percentage by which the harmonics are tempered in relation to the temperaments and tuning chosen. When this control is set to 100%, the harmonics will be fully tempered to provide maximal sensory consonance; when set to 50%, the harmonics will be tempered halfway towards maximal consonance; when set to 0% no harmonic tempering will occur. The percentage values assume that pitch is measured according to a logarithm of frequency, and is applied by tempering the generating harmonics by the appropriate percentage.

This control allows for the artist to determine the trade-off between sensory consonance and coherence. It also has other valuable uses – in many cases it may be found preferable to set this control to a value slightly short of 100%, so that slow but pleasant beating occurs; and to lower values still, to ensure that individual notes of consonant chords do not blend so strongly that they lose their sonic independence.

3.4 X_Spectra and Purer-Tuning

When the X_Spectral harmonic retuning is 100% engaged, it provides maximal sensory consonance regardless of the tuning chosen. This means that the Purer-Tuning range (discussed in Section 2.4.4) now refers to the tunings of the harmonics not of the primary intervals; so, within the Purer-Tuning range, all of the harmonics associated with the selected X_Temperament can be purely tuned. When X_Spectra is fully engaged, therefore, Purer-Tuning becomes more a measure of coherence than of consonance.

4 X_Timbre

An X_Timbre is a timbre modified by *primeness*, *conicality*, and *richness*.

In the same way that it is possible to manipulate the tunings of harmonics so that they relate to a temperament, it is also possible to alter their amplitude to affect the steady-state timbre of the sound in a way that affects both its character and the way that it interacts with the X_Temperament it is using.

4.1 Primeness

The prime factorisations of the harmonic numbers, shown in Table 2, indicate how every harmonic can be understood to be a product of integral powers of various primes. The prime factorisation theorem tells us that every integer has a *unique* prime factorisation.

Harmonics 2, 4, 8, 16, ..., 2^n are factorised only by prime 2, and so these harmonics can be said to embody “twoness”. Harmonics 3, 9, 27, ..., 3^n are factorised only by prime 3, and so can be said to embody “threeness”. Harmonics 5, 25, 125, ..., 5^n are factorised only by prime 5, and so can be said to embody “fiveness”.

Other harmonics are factorised by two, or more, different primes. Harmonic 12 is factorised by both 2 and 3, and so embodies both twoness and threeness; harmonic 15 is factorised by 3 and 5, and so embodies both threeness and fiveness.

The aim of the X_Spectral system is to allow the user to manipulate the spectral content such that the twoness, threeness, fiveness, ..., *primeness* of any given sound can be enhanced or reduced.

This allows for a timbre to be adjusted so that it brings out the salient qualities of any given temperament. For instance, in a temperament suitable for 7-limit harmony (such as the meantone $T = +4F$, $D = +10F$ temperament, where $F \approx 696.9c$), the artist can highlight the 7-limit consonances by “turning up” the sevenness. In a 3-limit temperament such as the medieval Pythagorean tuning, the restlessness of the ditones can be moderated by lessening the 5-limit aspects of the timbre – i.e. by turning down the fiveness. In a system which uses 6:7:9 chords as its primary consonances, the fiveness can be reduced, and the threeness and sevenness increased.

In the context of an X_Spectrum, where the relevant consonances are dynamically optimised to relate to the tuning, the manipulation of primeness can be used to highlight those consonances which are implied by the chosen temperament, *regardless* of the actual tuning chosen. Indeed, the manipulation of primeness becomes a way to control which types of chord are heard as consonant and which are heard as dissonant.

The manipulation of primeness also allows for simple but profound timbral manipulation – a means of changing the “character” of sounds with the minimum of technical knowledge. With the number of parameters limited to as few as three elements – twoness, threeness, and fiveness – radical timbral manipulation can still be achieved. For instance, fully turning down twoness will lead to an odd-harmonic-only timbre – a “hollow or nasal” sound (Helmholtz 1954, p.119) reminiscent of cylindrical closed bore instruments (e.g. clarinet); as the twoness is turned up, the even harmonics are gradually introduced creating a sound more reminiscent of open cylindrical bore instruments (e.g. flute, shakuhachi), or conical bore instruments (e.g.

bassoon, oboe, saxophone). This perceptual feature is called the *odd/even relation* or *conicality*. Turning up or down other primenesses will give different but recognisable qualitative changes. With the addition of controls for seven and higher primenesses, even more complex timbral changes can be undertaken, for instance enhancing the higher primenesses will create brighter, noisier, more complex timbres.

The algorithm works by making a set number of primeness gain controls available to the artist. Within the algorithm, the primeness gains g_p are scaled from a minimum of 0 to a maximum of 2 (these can be converted to decibels using $20 \log(g)$, and so represent the gain range of $-\infty$ dB to +6 dB). Each harmonic (or tempered harmonic) has its current amplitude multiplied by the minimum value of all of the primeness gains that correspond to its prime factorisation. This final gain applied to each harmonic is g_k (where k is the harmonic index). The reason that the minimum value is chosen is because the amplitude of beating is proportional to the minimum amplitude of the sinusoids involved (Sethares 2004, p.359). This means that the amplitude of the primeness being controlled is always proportional to the level of sensory dissonance, and therefore the relative sensory consonance, of the prime-limit intervals it is being used to control.

An additional meta-parameter called *richness* is also included. The richness control automates the process of progressively fading in higher and higher primenesses. When the richness control is at minimum, only the fundamental sounds; as it is increased, the twoness gain is increased, then the threeness gain, then the fiveness gain, etc.. The richness gains are applied after the individual primeness gains to ensure that when richness is at its maximum level the individual primeness gains remain unaltered.

Other timbral “dimensions” such as *brightness*, *tristimulus* levels, and *irregularity* have no direct connection to the X_System, but can still be added to provide further means of spectral control. Spectral manipulation over time, such as temporal enveloping and shimmer, is another vital component of timbre, but is beyond the scope of this paper.

By using gains rather than amplitudes for the primenesses, it can be seen that the X_Timbre is designed to operate as a modifier of a pre-existing timbre. Bill Sethares has developed an algorithm to allow for X_Spectral retuning and X_Timbral gain changes to be applied in real-time to an audio stream, providing that that stream is monophonic and contains meta-data indicating the fundamental pitch of every note played. This meta data can be supplied by any MIDI hardware or software synthesiser, as long as all chordal notes are sent on separate channels. This means that subtractive, FM, or physical modelling, etc., synthesis can provide the initial sound source and yet still be subject to the unique level of precise control that is usually only found in the domain of additive synthesis. Since the algorithm uses re-synthesis, which separates noise from harmonic components, there are none of the noise problems usually associated with analogue amplification.

Alternatively, the X_Timbre algorithm can be utilised as a pure additive synthesis technique by generating sine waves of harmonic index k with amplitude a_k , such that $a_k = g_k k^{-1}$. This equation ensures that when all gains are set to 1, the resulting waveform is a sawtooth – the conventional starting point for subtractive synthesis and so a familiar starting point for further timbral modification.

5 X_Tonality and X_Scales

X_Tonality is the theory of tonal functionality for p -limit just intonations, and the temperaments derived from them. X_Tonality is required to derive any p -limit system's X_Scales.

X_Scales are the receptacles of tonal function within any given temperament. They supply sufficient but not redundant or superfluous resources for melody, harmony, and most importantly tonality. Their derivation, within any given temperament, allows for that temperament's tonal tuning boundaries to be determined – these are the points beyond which that temperament's tonal functionality will fail.

The work for deriving the X_Tonality for non-meantone and higher than 5-limit systems is ongoing, so what follows is a brief outline of the theory for 5-limit meantone.

5.1 Melodic Resource

For a scale to sound like a coherent entity that has had nothing removed from it or anything extraneous added to it, it must have *constant structure* or be perceivable as a having an *altered constant structure*.

Definition: *Constant structure* is a term used by Erv Wilson to indicate a scale in which all intervals (as measured by size) span the same number of notes (Grady).

Constant structure scales are heard as being internally consistent, logically organised, and therefore complete and self-contained, because any addition or removal of a single note will (with a few exceptions) break the constant structure. All MOS scales have constant structure.

Scales that do not have constant structure can still be perceived as complete if they are heard to be an alteration of a constant structure scale. For example, both the harmonic minor scale and the (ascending) melodic minor scale can be heard as alterations of the diatonic (or natural) minor scale. Other non-constant structure scales, such as the diatonic hexachord, cannot be heard as being an alteration of a different constant structure scale.

A non-constant structure scale can only be heard as an *altered constant structure scale* if it cannot be made to have constant structure by the addition or removal of one note.

For example, the harmonic and (ascending) melodic minor scales cannot be given constant structure through the addition or removal of a single note; conversely, the diatonic hexachord (1, 2, 3, 4, 5, 6) can – the simple addition of 7, or the removal of 4 will create constant structure scales (the diatonic and pentatonic scales, respectively).

A scale which has neither constant structure, nor altered constant structure, will be heard as if a note has been removed or added to an otherwise complete and coherent structure. This means that, for melodic purposes, it is either insufficient or superfluous.

5.2 Harmonic Resource

For a scale to function as an effective harmonic resource, it must provide sufficient but not superfluous resources for the prime major and minor chords. This requires that

every member of the scale is also a member of at least one complete prime chord. If a note is not a part of a complete prime chord it is superfluous to the purpose of harmony.

Assuming that the scale is built from an *s-chain* (like an MOS scale), then the minimum number of notes required to fulfil the above *harmonic condition* is dependent on the temperament being used.

For example, the $T = +4F$ (meantone) temperament requires a minimum of 6 notes to fulfil the harmonic condition; the $F = +5T$ temperament requires a minimum of 8 notes; the $F = -6S$ temperament requires 10 notes.

Figure 25 shows the intersection of MOS scales, regular temperaments and the harmonic condition. The red “temperament lines” are placed at the optimal tuning for each temperament; the ring from which each temperament line starts indicates the number of notes required to fulfil the harmonic condition. This means that the first MOS scale (as indicated by the blue arcs) that the temperament line passes through is the lowest cardinality MOS scale that can fulfil the harmonic condition for that regular temperament.

For example, the 7 note diatonic scale is the lowest cardinality MOS scale that fulfils the harmonic condition when using the $T = +4F$ (meantone) temperament; 10 notes is the lowest cardinality MOS scale that fulfils the harmonic condition when using the $F = +5T$ (magic) temperament; 11 notes is the lowest cardinality MOS scale that fulfils the harmonic condition when using the $F = -6S$ (Hanson) temperament.

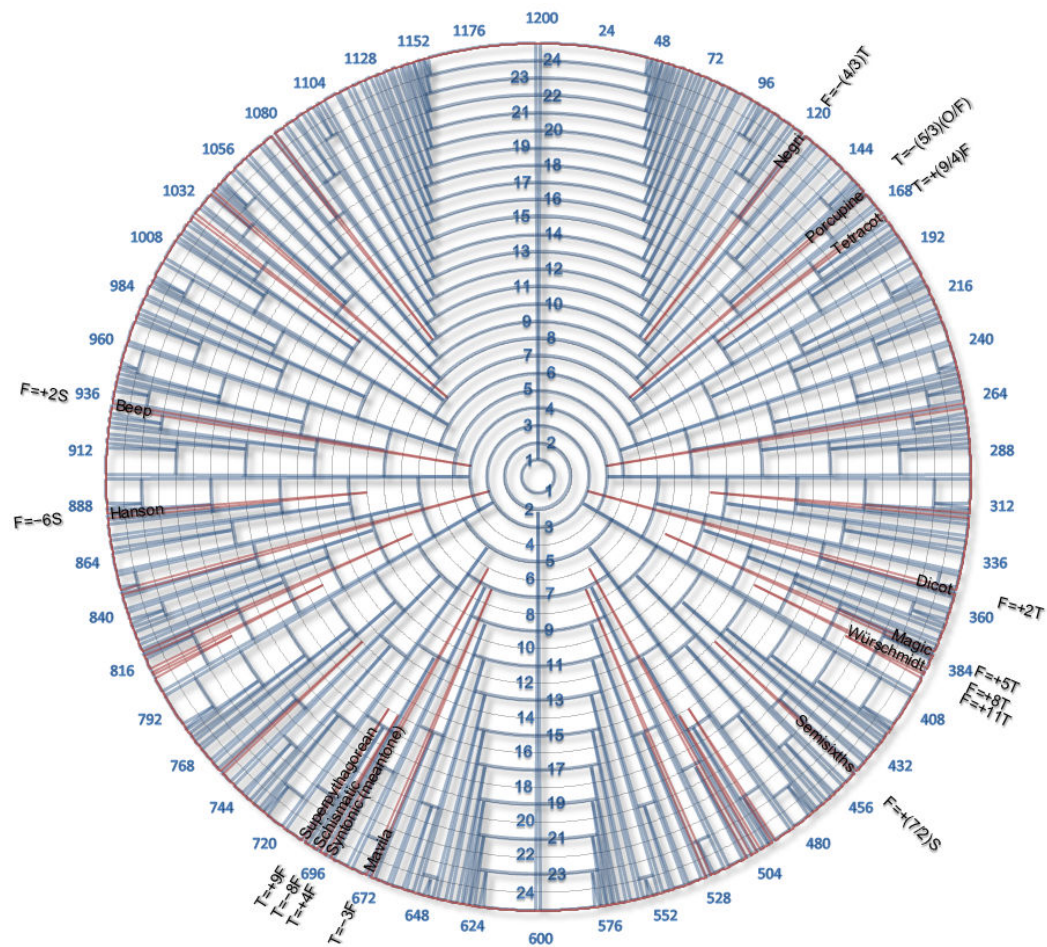


Figure 25. MOS scales, regular temperaments and the harmonic condition.

5.3 Tonal Resource

For a scale to function as a tonal resource it must have a coherent tonal functionality that is capable of giving a tonic function to one, or more, of its prime chords.

Devoid of any harmonic context: any melodic interval that is a semitone (or so) from a unison will be heard as an alteration of that unison; similarly, any interval that is a semitone (or so) from a perfect fifth or fourth will also be heard as an alteration of that perfect fifth or fourth – with the exception of the major third-like and minor sixth-like intervals, because these have some measure of inherent stability.

So, devoid of harmonic context, semitone-like intervals are heard as alterations of the *prototypical* unison; tritone-like intervals are heard as alterations of the prototypical perfect fifth/perfect fourth.

In 5-limit harmony, the major (4:5:6) and minor (10:12:15) triads are the prime chords.

The major and minor triads that share the same root are said to be parallel triads (e.g. C-major and c-minor). The third of these two triads differ by a semitone, but their root remains the same. Because of this, a major triad has the capacity to be heard as an

alteration of a minor triad and a minor triad has the capacity to be heard as an alteration of a major triad.

Between any pair of triads there are a total of nine melodic relations (root of first triad with root of second triad, root of first triad with third of second triad, root of first triad with fifth of second triad, etc).

Calculating, for all possible triad pairings, the number of melodic relations that are unisons/octaves and perfect fifth/fourths (prototypes) minus the number of semitones and tritones (alterations), shows that there are *seven* different triad pairings that have a surplus of prototypical intervals compared to alterations them. These seven pairings are heard as *prototypical pairings* and all non-prototypical pairings are heard as alterations of these.

For example, the non-prototypical IV \rightarrow V progression is heard as an alteration of the prototypical IV \rightarrow v progression.

The scale with the highest cardinality that contains only prototypical pairings is the diatonic hexachord.

Any scale that differs from the diatonic hexachord is heard as an alteration of the hexachord and the altered notes seek resolution to the next available hexachordal note in the same direction as the alteration. This is the origin of tonal function, and explains all the tonal cadences that can be found in 5-limit harmony.

This is all explained in greater depth in Milne (2005).

5.4 X_Scales

The X_Scales are those scales that function simultaneously as an effective melodic, harmonic, and tonal resource.

In 5-limit harmony they are (their tonal modes are shown in brackets):

- Diatonic (Major – 1, 2, 3, 4, 5, 6, 7); (Minor – 1, 2, $\flat 3$, 4, 5, $\flat 6$, $\flat 7$)
- Harmonic Minor (1, 2, $\flat 3$, 4, 5, $\flat 6$, 7)
- Harmonic Major (1, 2, 3, 4, 5, $\flat 6$, 7)
- Melodic (Minor – 1, 2, $\flat 3$, 4, 5, 6, 7); (Major – 1, 2, 3, 4, 5, $\flat 6$, $\flat 7$)
- Double Harmonic (Major – 1, $\flat 2$, 3, 4, 5, $\flat 6$, 7); (Minor – 1, 2, $\flat 3$, $\sharp 4$, 5, $\flat 6$, 7).

5.5 Tuning Ranges and X_Tonality

The derivation of tonal function allows for strict limits to be placed on the tuning ranges within which tonal function can exist, regardless of X_Spectral manipulation. In 5-limit harmony using $T = +4F$ those limits are determined by MOS:12 – as the tuning approaches 5-tet, tonal function disappears completely; as it approaches 7-tet, it becomes ambiguous and illusory.

6 X_Layout

An X_Layout ensures that, within any given X_Temperament, the note layout will be isomorphic. This means that the same interval will always have the same shape wherever it is played on the button-field and whatever X_Tuning it is using.

6.1 Lattices and Vector Generators

For a button-field to be capable of isomorphism, it must be intrinsically isomorphic in itself – i.e. it must have discrete translational symmetry. A structure that has translational symmetry is called a *lattice*. There are different types of 2-dimensional lattice, e.g. square, hexagonal, parallelogrammic, but they can all be thought of as square lattices with successive rows staggered a fraction of a step to the right or left.

In the square lattice, let the unit vectors (1, 0) and (0, 1) represent the distance between contiguous notes horizontally and vertically, respectively. In the hexagonal lattice (as used on the Thummer), because every alternate row is staggered half a button step to the right, valid vectors are of the form (m, 2n) and (m – 0.5, 2n – 1), where m and n are positive or negative integers.

Definition: Two or more vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$, are *linearly dependent* if there are integers z_1, z_2, \dots, z_n , not all zero such that $z_1 \mathbf{v}_1 + z_2 \mathbf{v}_2 + \dots + z_n \mathbf{v}_n = \mathbf{0}$. If there are no such z_1, z_2, \dots, z_n , then $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$, are said to be *linearly independent*.

If the *multiplicatively independent* generating intervals of the tuning system are mapped to *linearly independent generating vectors*, the rank of the tuning system and the dimension of the lattice are the same; the mapping is said to be *isomorphic* (i.e., having the same form). In particular, a tuning system with rank greater than two cannot be isomorphically mapped to a 2-dimensional lattice. Nor can a rank-2 system be isomorphically mapped to a 1-dimensional keyboard lattice.⁶

Accordingly, the discussion is now restricted to rank-2 tuning systems, which can be described with two vectors. The repetition vector corresponding to the unit of repetition $s(U)$ is notated $\mathbf{u} = (u_x, u_y)$; the generating vector corresponding to the generating interval $s(G)$ is notated $\mathbf{g} = (g_x, g_y)$.

In order to ensure that every button has an interval mapped to it, it is necessary that every member of the lattice can be derived as an addition of integral multiples of the generating vectors. For this to be the case:

$$\mathbf{u} \times \mathbf{g} = \pm 1, \text{ hence} \quad (3)$$

$$u_x g_y - u_y g_x = \pm 1 \quad (4)$$

The generating set chosen by the algorithm in Section 2.3.1 is ideally suited for a direct mapping to the button lattice, because it consists of simple and familiar intervals that quickly generate the derived consonances defined by any given temperament. For a finitely sized lattice, it is necessary to ensure that the generating vectors are chosen

⁶ To the extent that the standard Halberstadt keyboard design is considered to be a 1-dimensional lattice, it is clear that this keyboard type is incapable of isomorphically mapping rank-2 temperaments; it can only isomorphically map rank-1 (i.e. equal) temperaments.

such that they allow for a chain of unit-reduced generators sufficiently long enough to produce a workable number of derived consonances.

The cardinality of the s -chain hosted on a button-field gives the number of notes available per unit of repetition. Since any instrument has a finite number of notes, this means that there is an inevitable trade-off between the number of unit-reduced intervals and the overall octave range.

The following section puts this trade-off onto a firmer mathematical basis.

6.2 Intervals versus Octaves

The vector \mathbf{c} of the n^{th} note in an s -chain is given by $\mathbf{c} = (n - 1)\mathbf{g} - \text{trunc}(\log_O(s^{n-1}))\mathbf{u}$, hence:

$$\begin{aligned} c_x &= (n - 1)g_x - \text{trunc}(\log_O(s^{n-1}))u_x \text{ and} \\ c_y &= (n - 1)g_y - \text{trunc}(\log_O(s^{n-1}))u_y \end{aligned} \quad (5)$$

The above formulae allow for the positions of successive notes in an s -chain to be plotted as they cut a *swathe* across the lattice. Figure 26 shows the 19-note F -chain swathe that is cut across the Thummer using its default mapping, and starting from C_1 :

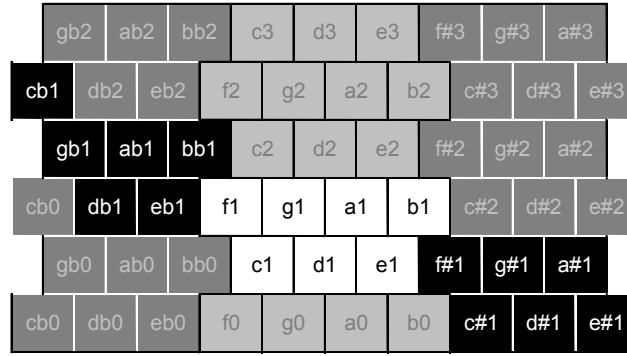


Figure 26. F -chain swathe on the Thummer using the default Wicki mapping.

The average gradient m , and average thickness T (as measured orthogonally to the gradient) of the s -chain swathe, is determined by \mathbf{u} , \mathbf{g} , and the exponent z that relates O to s . Given $s = O^z$:

$$m = (g_y - u_y z) / (g_x - u_x z) \quad (6)$$

$$T = \left| (1/z) / (((g_x/z) - u_x)^2 + ((g_y/z) - u_y)^2)^{1/2} \right| \quad (7)$$

The default mapping of the Thummer uses $\mathbf{u} = (0, 2)$, $\mathbf{g} = (0.5, 1)$, which gives the following values for gradient and thickness when $s = O^{7/12}$: $m = -1/3$; $T = 1.916$.

The swathe thickness T gives a direct measure of the extent to which any given mapping (i.e. choice of vectors \mathbf{u} and \mathbf{g}) favours the number of octave reduced intervals compared to the number of octave repetitions of those intervals, over the complete range of possible s -tunings. When T is large, the s -chain cuts a wide swathe through the lattice and so uses up more of its space for unit-reduced intervals and leaves less space for octave repetitions of them. When T is small, the s -chain cuts a

narrow swathe through the lattice uses up less lattice space for unit-reduced intervals, and so leaves more lattice space for octave repetitions of them.

6.2.1 Rotational Families

The above formulae for $\mathbf{u} \times \mathbf{g}$, m , and T show that:

- Swapping the value of g_x with g_y and swapping the value of u_x with u_y , has no effect on the T -curve, but it flips the sign of $\mathbf{u} \times \mathbf{g}$, and inverts the swathe gradient.
- Flipping the signs of u_x and g_x reflects the swathe about the y-axis; flipping the signs of u_y and g_y reflects the swathe in the x-axis. Neither operation has any effect on the T -curve, but they flip the sign of $\mathbf{u} \times \mathbf{g}$ and the gradient m .
- Flipping the signs of g_x and g_y and u_x and u_y , has no effect on the T -curve, $\mathbf{u} \times \mathbf{g}$ or m .

The above described swaps and flips, place the different \mathbf{u}, \mathbf{g} mappings into eight-member *rotational families* with identical T -curves, but four different gradients. The swap and flip operations are, therefore, a useful means to rotate any given swathe without affecting its thickness.

It also means that it is possible to use only $\mathbf{u} \times \mathbf{g} = 1$ to determine valid g_x, g_y, u_x, u_y values – because the values that give $\mathbf{u} \times \mathbf{g} = -1$ will give identical T -curves and are really just rotations of the swathe. This makes the determination of valid values easier, and allows the whole rotational family to be represented by just one member.

The rotational family of mappings with the same T -curve as the default Thummer layout are:

- $\mathbf{u} = (0, 2), \mathbf{g} = (0.5, 1) : m = -1/3$
- $\mathbf{u} = (0, 2), \mathbf{g} = (-0.5, 1) : m = 1/3$
- $\mathbf{u} = (0, -2), \mathbf{g} = (0.5, -1) : m = 1/3$
- $\mathbf{u} = (0, -2), \mathbf{g} = (-0.5, -1) : m = -1/3$
- $\mathbf{u} = (2, 0), \mathbf{g} = (1, 0.5) : m = -3$
- $\mathbf{u} = (-2, 0), \mathbf{g} = (-1, 0.5) : m = 3$
- $\mathbf{u} = (2, 0), \mathbf{g} = (1, -0.5) : m = 3$
- $\mathbf{u} = (-2, 0), \mathbf{g} = (-1, -0.5) : m = -3$

6.2.2 z-cluster Families

The derivative of T , with respect to z , is solvable for values of z when the derivative equals zero. This value of z is called z^* , and gives the value of z where T is at its maximum. The value of T at z^* is called T_{max} , the value of m at z^* is m^* :

$$z^* = (g_x u_x + g_y u_y) / (u_x^2 + u_y^2) \quad (8)$$

Note that when $u_x = 0$, $z^* = g_y / u_y$.

Substituting z^* into the above equations for T and m , and given $\mathbf{u} \times \mathbf{g} = \pm 1$

$$T_{max} = (u_x^2 + u_y^2)^{1/2} = \|\mathbf{u}\| \quad (9)$$

$$m^* = -u_x / u_y \quad (10)$$

This shows that T_{max} is equivalent to the length of the repetition vector, and that the swathe at z^* travels orthogonally to \mathbf{u} .

The second derivative of T , with respect to z , is solvable when $z = z^*$. This curvature value $T_{curvature}$ indicates the curvature of the T -curve at z^* – a low value indicates a wide gentle peak, a large value indicates a spiky peak.

$$T_{curvature} = -(u_x^2 + u_y^2)^{5/2} / (u_x g_y - u_y g_x)^3 \quad (11)$$

Given $\mathbf{u} \times \mathbf{g} = \pm 1$,

$$\begin{aligned} T_{curvature} &= \pm (u_x^2 + u_y^2)^{5/2} \\ &= \pm T_{max}^5 \end{aligned} \quad (12)$$

This shows that the higher the value of T_{max} , the greater T 's curvature, and so the more specific the mapping is to a smaller tuning range of generator intervals. A mapping, therefore, can only produce a wide swathe over a small tuning range.

Given $\mathbf{u} \times \mathbf{g} = \pm 1$, all \mathbf{u}, \mathbf{g} mappings that share the same values for u_y and g_y , have z^* values that cluster symmetrically about g_y / u_y , and so constitute a *z-cluster family* of mappings.

Given $\mathbf{u} \times \mathbf{g} = \pm 1$, the derivative of z^* with respect to u_x shows that the minimum, central, and maximum z^* values are found when $u_x = u_y$, $u_x = 0$, and $u_x = -u_y$, respectively:

- when $u_x = u_y$, $z^* = (g_x + g_y) / 2 u_y$;
- when $u_x = 0$, $z^* = g_y / u_y$;
- when $u_x = -u_y$, $z^* = (g_y - g_x) / 2 u_y$;
- as $u_x \rightarrow \infty$, $z^* \rightarrow g_y / u_y$.

Given $u_{x1}, u_{y1}, g_{x1}, g_{y1}$, the following transformation to $u_{x2}, u_{y2}, g_{x2}, g_{y2}$: keeps the same T_{max} value; reflects z^* about $z = g_{y1} / u_{y1}$; flips the sign of $\mathbf{u} \times \mathbf{g}$; keeps the same gradient.

- $u_{x2} = u_{x1}$
- $u_{y2} = u_{y1}$
- $g_{y2} = g_{y1}$
- $g_{x2} = (2 u_{x1} g_{y1} / u_{y1}) - g_{x1}$ (this ensures $\mathbf{u} \times \mathbf{g} = \pm 1$)

Given $u_{x1}, u_{y1}, g_{x1}, g_{y1}$, the following transformation to $u_{x2}, u_{y2}, g_{x2}, g_{y2}$: keeps the same T_{max} value; reflects z^* about $z = 1/2$; flips the sign of $\mathbf{u} \times \mathbf{g}$; keeps the same gradient.

- $u_{y2} = u_{y1}$;
- $g_{y2} = u_{y1} - g_{y1}$ (the above two conditions are necessary to ensure that $g_{y1} / u_{y1} + g_{y2} / u_{y2} = 1$, which is necessary to ensure that the z^* values for $\mathbf{u}_1, \mathbf{g}_1$ and $\mathbf{u}_2, \mathbf{g}_2$ are reflected about $z = 1/2$);

- $u_{x2} = u_{x1}$ (given that $u_{y2} = u_{y1}$, this ensures that T_{max} remains the same);
- $g_{x2} = u_{x1} - g_{x1}$ (this ensures that $\mathbf{u} \times \mathbf{g} = \pm 1$, and it flips its sign).

The above formulae allow the engineer to make some quick rule-of-thumb decisions about the mapping – T_{max} is equivalent to the length of \mathbf{u} , and z^* is approximately equivalent to g_y / u_y . The remaining variable g_x is then determined by the other three, such that $\mathbf{u} \times \mathbf{g} = \pm 1$.

Figure 27 shows the T -curves for a series of mappings (with $O = 2$). Notice how the mappings have $g_y / u_y = 2/3$ or $1/3$, and so cluster at $2^{2/3}$ (800 cents) and $2^{1/3}$ (400 cents). The remaining parameters are chosen to ensure that every T curve has an exact reflection about $z = 1/2$ (i.e. $2^{1/2}$, or 600 cents).

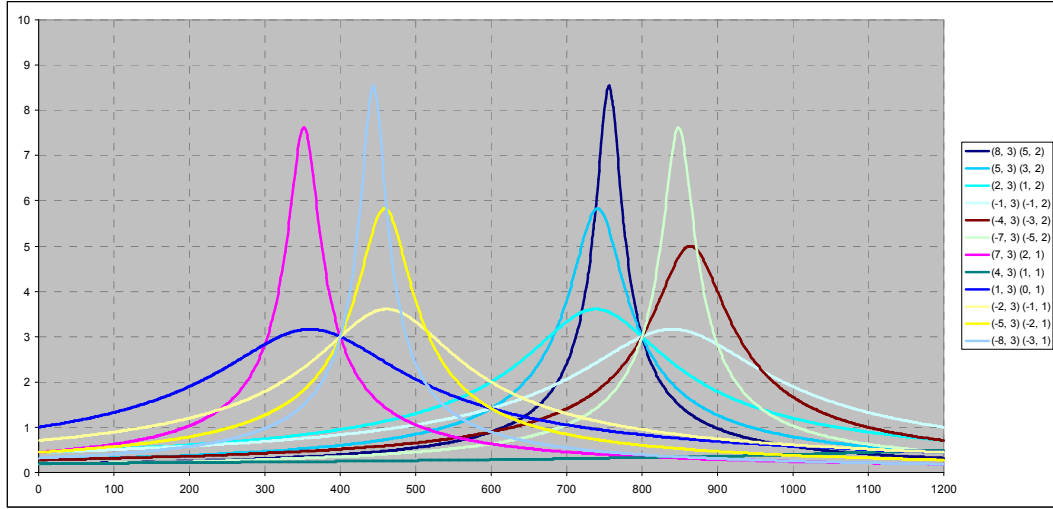


Figure 27. T -curves of various mappings reflected about 600 cents.

6.3 Optimising Mappings

This section discusses methods by which mappings of generators to lattice vectors can be optimised to meet certain criteria.

6.3.1 T -Curves

As discussed above, T indicates the balance a mapping gives to the number of octave-reduced intervals compared to the number of octave repetitions of those intervals. This balance is a critical engineering trade-off. Although the final balance depends on the size, shape, and orientation of the button lattice, the underlying shape of the T curve has fundamental implications on what any lattice is realistically capable of delivering over the range of s -tunings for which it is intended to be used.

An ideal mapping produces a swathe thickness over the range of s -tunings it is intended to host. That thickness certainly needs to be greater than one.

This can be demonstrated by comparing the T -curve of the Fokker mapping (Fokker) with the T -curve of the Wicki/Hayden mapping. The Fokker mapping uses $\mathbf{u} = (6, 2)$

and $\mathbf{g} = (3.5, 1)$, the Wicki mapping uses $\mathbf{u} = (0, 2)$ and $\mathbf{g} = (-0.5, 1)$. Figure 28 shows plots of their T -curves over the tuning range from 0 to 1200 cents.⁷

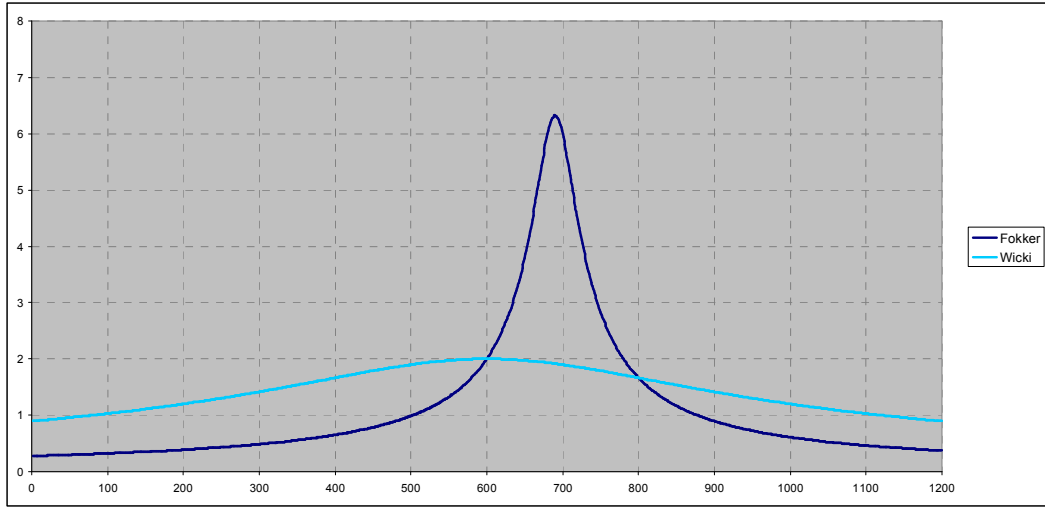


Figure 28. A comparison of the T -curves produced by the Fokker and Wicki mappings.

Around $s = 700$ cents, the Fokker mapping gives a very wide swathe – so wide that it swamps a smaller button-field, and requires a lot of buttons to enable a reasonable octave range. The Wicki mapping provides a less wide swathe and so has a better octave range, though a smaller number of octave-reduced intervals, which makes it more suited to general use, though perhaps less suited for deeper microtonal experimentation.

The following two figures show how the s -chain swathe generated by the two different mappings cut through the Thummer's button-field when $s = O^{7/12}$. The Fokker mapping provides a high cardinality s -chain of 36, but an octave range of only 1.583 ($T = 6$, $m = \infty$):

⁷ Note that the Bosanquet mapping described by Ellis in Helmholtz (1877 p.479), and the identical Erv Wilson mapping (Wilson) are to a non-hexagonal parallelogrammic lattice, but are almost identical to an x -axis reflection of the Fokker mapping. All comments about the Fokker mapping, therefore, are also applicable in a general sense to the Bosanquet/Wilson mapping. The Janko mapping is only specified for 12-tet, in which case it is identical to the Fokker mapping.

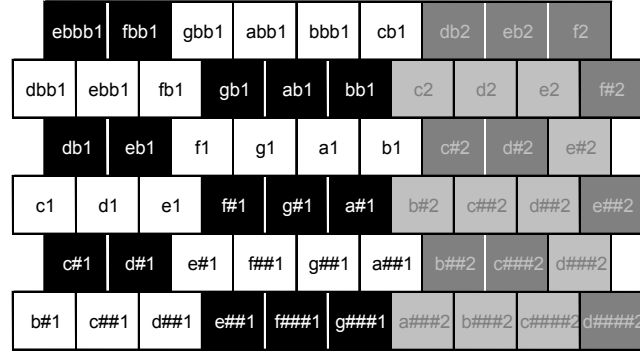


Figure 29. The F -chain swathe produced by the Fokker mapping when using $F = O^{7/12}$

The Wicki mapping gives a lower cardinality s -chain of 19, but a larger octave range of 3 ($T = 1.897$, $m = -0.333$):

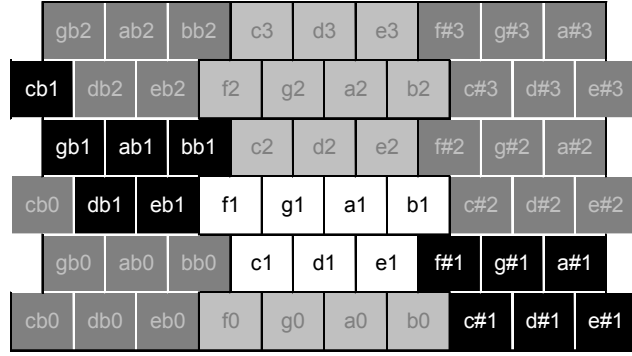


Figure 30. The F -chain swathe produced by the Wicki mapping when using $s = O^{7/12}$.

The principal issue with the Fokker mapping, however, occurs when it hosts a T -chain tuning system (where $s \approx 386$ cents) or an S -chain tuning system (where $s \approx 884$ cents). As can be seen in Figure 28, at these two s -tunings the Fokker swathe thickness drops to below 1, which makes it completely unsuitable for temperaments that require such tuning systems; in comparison, the Wicki mapping still has a reasonably good thickness.

For example, choosing an s -tuning of approx $5/4$ (the tuning range suitable for temperaments such as $F = +5T$), the Fokker mapping onto the Thummer button-field provides a T -chain with a wholly inadequate 3 notes; the Wicki mapping provides a T -chain of 15 notes. With a chain of only three major thirds, the Fokker mapping to the Thummer button-field would be unable to supply any tempered 3-limit consonances, whereas the Wicki mapping will supply an ample number.

Similarly, choosing an s -tuning of approx $5/3$ (the range suitable for temperaments such as $F = -6S$), the Fokker mapping onto the Thummer button-field provides an S -chain of only 7 notes; the Wicki mapping provides an S -chain of 14 notes.

Figure 31 shows the T -curves for all possible mappings that fit onto an hexagonal lattice, with T_{max} no higher than eight. The highlighted T -curves are those for the

mappings with some historical precedence – in order of height they are: Wesley; Wicki; CBA (Chromatic Button Accordion); Fokker.⁸

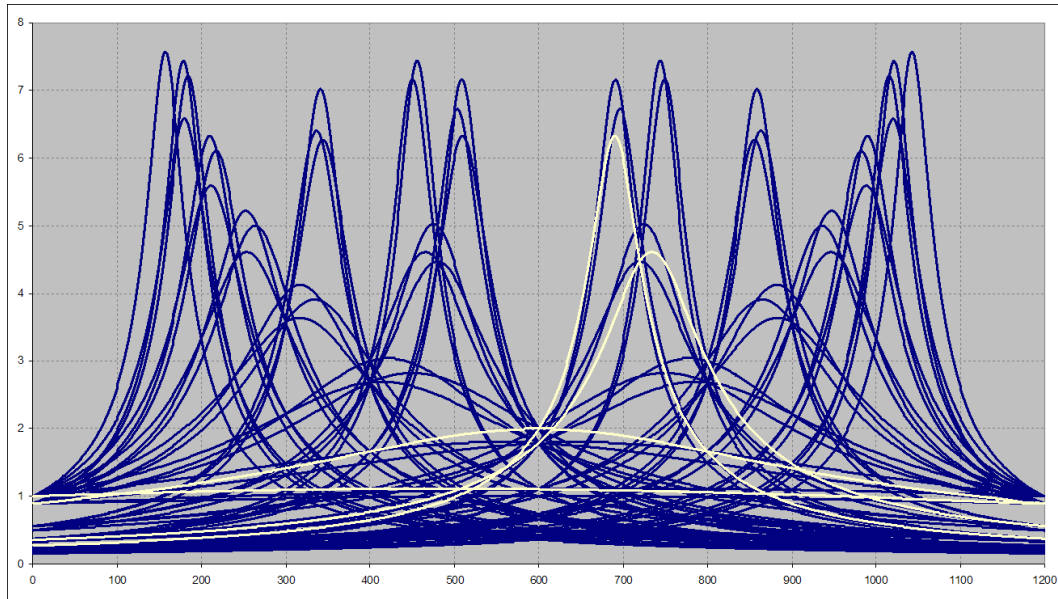


Figure 31. The T -curves of all possible mappings to an hexagonal lattice that have $T_{max} < 8$. Wesley, Wicki, CBA, and Fokker mappings are highlighted.

Figure 31 indicates that there is a choice between using:

- one *generalist mapping* for all possible s -chain tuning systems;
- *specialised mappings* that focus on smaller s -tuning ranges – for example, it is possible to have a specialist mapping for F -chain tuning systems, another specialist mapping for T -chain tuning systems, another specialist mapping for S -chain tuning systems.

If the choice is for one generalist mapping, then the Wicki mapping is the clear winner – it is centred at the centre of the s -tuning range (600 cents) and has a broad sweep across the spectrum, falling below one only at the very extremes of the s -tuning range.

If the choice is for specialist mappings, a selection of mappings such as those shown in Table 3 and Figure 32 could be used:

⁸ Note that the Wesley mapping is only specified for 12-tet, but can be recast for all possible generator tunings. The Wesley mapping has $\mathbf{u} = (0.5, 1)$, $\mathbf{g} = (1, 0)$; Wicki has $\mathbf{u} = (0, 2)$, $\mathbf{g} = (0.5, 1)$; CBA-C has $\mathbf{u} = (3.5, -3)$, $\mathbf{g} = (2, -2)$; CBA-B has $\mathbf{u} = (3.5, 3)$, $\mathbf{g} = (2, 2)$; Fokker has $\mathbf{u} = (6, 2)$, $\mathbf{g} = (3.5, 1)$.

Table 3. A selection of twelve specialist mappings suitable for F, T and S-chain tuning systems.

Tuning system	u_x, u_y	g_x, g_y	T_{max}	Z^* (cents)
F-chain:	3.5, 3	2, 2	4.6	734
	-0.5, 5	-0.5, 3	5.0	725
	-6, 2	-3.5, 1	6.3	690
	4.5, 5	2.5, 3	6.7	696
	-1.5, 7	-1, 4	7.2	691
	-4, 2	-2.5, 1	4.5	720
T-chain:	2.5, 1	1, 0	2.7	414
	2, 2	0.5, 1	2.8	450
	-0.5, 3	-0.5, 1	3.0	422
S-chain:	3.5, 1	2.5, 1	3.6	883
	-2.5, 3	-2, 2	3.9	866
	1, 4	0.5, 3	5.0	936

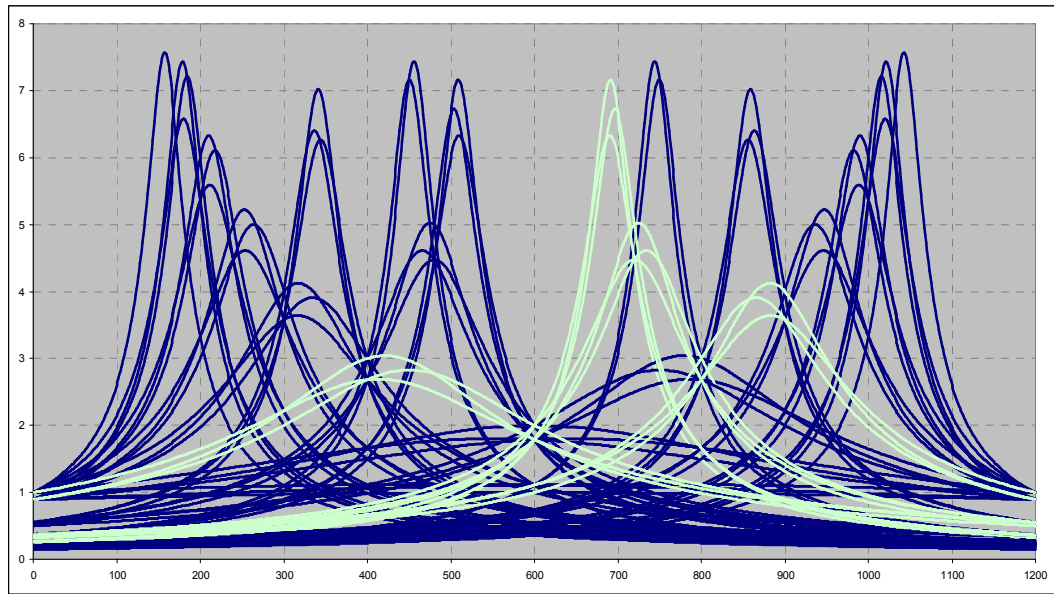


Figure 32. A selection of twelve possible mappings suitable for T, F and S-chain tuning systems.

Alternatively, two semi-generalised mappings could be used – one for s-tunings less than 600 cents one for s-tunings greater than 600 cents. Suitable mappings are shown in Table 4 and highlighted in Figure 33:

Table 4. A selection of mappings clustered around 400 and 800 cents.

Tuning system	u_x, u_y	g_x, g_y	T_{max}	z^* (cents)
800 cents:	2.5, 1	1.5, 1	2.7	786
	-2, 2	-1.5, 1	2.8	750
	0.5, 3	0, 2	3.0	778
400 cents:	2.5, 1	1, 0	2.7	414
	2, 2	0.5, 1	2.8	450
	-0.5, 3	-0.5, 1	3.0	422

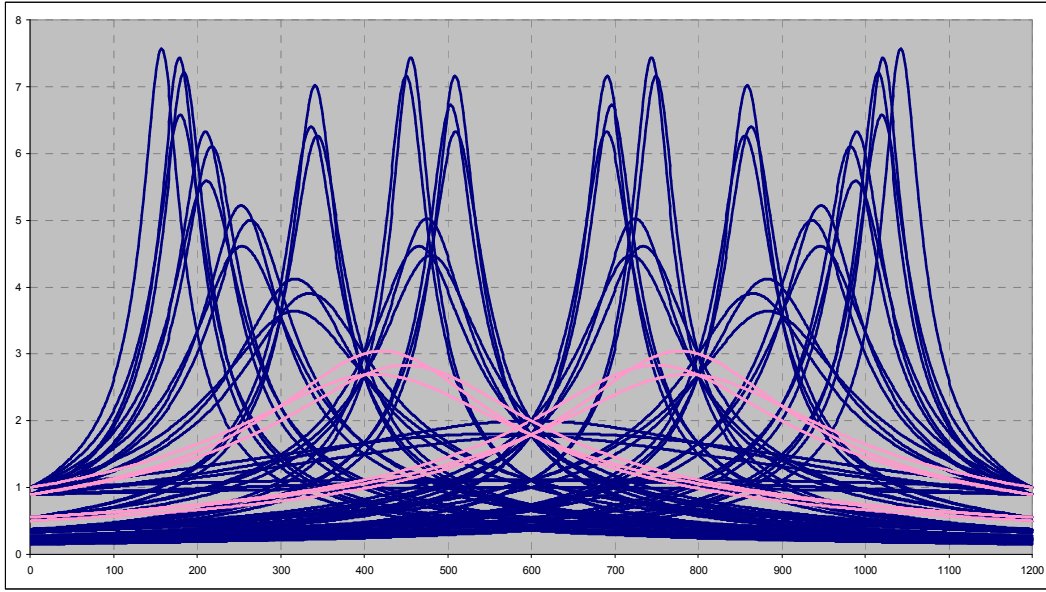


Figure 33. A selection of mappings clustered around 400 and 800 cents.

The mappings with high $||\mathbf{u}||$ values have a wider swathe over a narrower range of s -tunings, and so are suitable when the artist wishes to explore a larger range of octave-reduced (and therefore microtonal) intervals (at the expense of losing overall octave range).

When different specialist mappings are being used for different parts of the s -tuning range, an appropriate mapping can be stored as part of the temperament preset and will be automatically switched when the preset is chosen. When the s -tuning is being changed with the controller, mappings can be automatically changed at appropriate locations.

6.3.2 Additional Optimisations

The means to determine swathe width over the s -tuning range (Section 6.2.2), and the techniques to rotate these swathes without affecting their T -curves (Section 6.2.1), provide the principal means of designing and determining suitable mappings. Given that more than one choice is sometimes available, further decisions must be made according to any other factors deemed to be useful, such as:

- proximity of fifths and thirds, to allow for conventional triads to be easily played (with two or even just one finger);
- easily “finger-able” major and minor seconds – these two intervals being the building blocks of conventional 3 and 5-limit scales;
- artistry – that undefinable something that makes a mapping feel “right”.

In order to aid these decisions it is useful to know which interval is represented by which lattice vector.

The interval $s^m O^n$ found between any two lattice notes connected by a vector \mathbf{l} , is given by:

$$s^m O^n = (s^{u_x l_y - u_y l_x} O^{l_x g_y - l_y g_x})^{1/(u_x g_y - u_y g_x)} ; \quad (13)$$

since $\mathbf{u} \times \mathbf{g} = \pm 1$,

$$s^m O^n = (s^{u_x l_y - u_y l_x} O^{l_x g_y - l_y g_x})^{u_x g_y - u_y g_x} . \quad (14)$$

This means that the vectors that connect contiguous buttons on the hexagonal button-field produce the following intervals:

- (1, 0): $s^m O^n = (s^{-u_y} O^{g_y})^{u_x g_y - u_y g_x}$
- (0, 2): $s^m O^n = (s^{2u_x} O^{-2g_x})^{u_x g_y - u_y g_x}$
- (0.5, 1): $s^m O^n = (s^{-0.5u_y + u_x} O^{0.5g_y - g_x})^{u_x g_y - u_y g_x}$
- (-0.5, 1): $s^m O^n = (s^{0.5u_y + u_x} O^{-0.5g_y - 0.5g_x})^{u_x g_y - u_y g_x}$

6.4 Note Layouts and Temperaments

When using a generalised mapping, every temperament will use the same lattice vectors for the repetition and generating intervals, but all other intervals will differ according to the temperament chosen. This is inevitable, since it is the derivation of generated intervals that actually defines a temperament.

But, within any given X_Temperament, the relative positions of consecutive scale notes (and therefore the fingering) of the designated MOS scale remains consistent within its valid tuning range. So, for example, the fingering of the 7 note diatonic scale remains the same within the MOS:7 valid tuning range of $O^{1/2} < F < O^{3/5}$, and the fingering of the 12 note chromatic scale remains the same within the MOS:12 valid tuning range of $O^{4/7} < F < O^{3/5}$.

Because the X_Temperament generators are mapped to button-lattice vectors in a consistent fashion, the button-lattice itself becomes a geometric representation of both the X_Temperament and its related X_Spectrum. In this way the playing surface of the instrument exposes the inner geometry of the music it is being used to play.

7 Conclusion

The X_System is the integrated system of X_Elements. The integration of these elements provides the following benefits:

- All regular temperaments (e.g. the ubiquitous “meantone temperament” $T = +4F$) are isomorphically mapped to the button-field so that every interval in that temperament always has the same shape.
- Every temperament can be dynamically retuned using a simple controller, opening up all sorts of creative avenues.
- Temperament presets automatically provide optimal tuning, recommended tuning ranges, a simple means of indicating how the primary consonances are derived, and optional specialised note-layouts.
- Within any given temperament, the fingering of any MOS scale of given cardinality remains consistent within its valid tuning range.
- Whatever tuning being used, the consonance of the temperament’s prime chords can be automatically maximised through dynamic spectral retuning. This means that disruptive commas, such as the syntonic comma, can be removed without paying a consonance penalty.
- Dynamic spectral retuning allows for temperaments to be tuned over a wider range than would otherwise be possible, opening up new creative possibilities.
- The timbre can be adjusted so that it either enhances or diminishes the salient features of the temperament being used. The same techniques also allow for intuitive changes to be made to the character of the timbre.
- A default keyboard mapping which can function over a wide range of tunings and provides a good range of both intervals and overall octave range.
- A choice of alternative mappings designed to give more octave-reduced intervals over smaller tuning ranges, allowing for more extensive microtonal experimentation.

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